Traces, Executions, and Schedulers, Coalgebraically

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and

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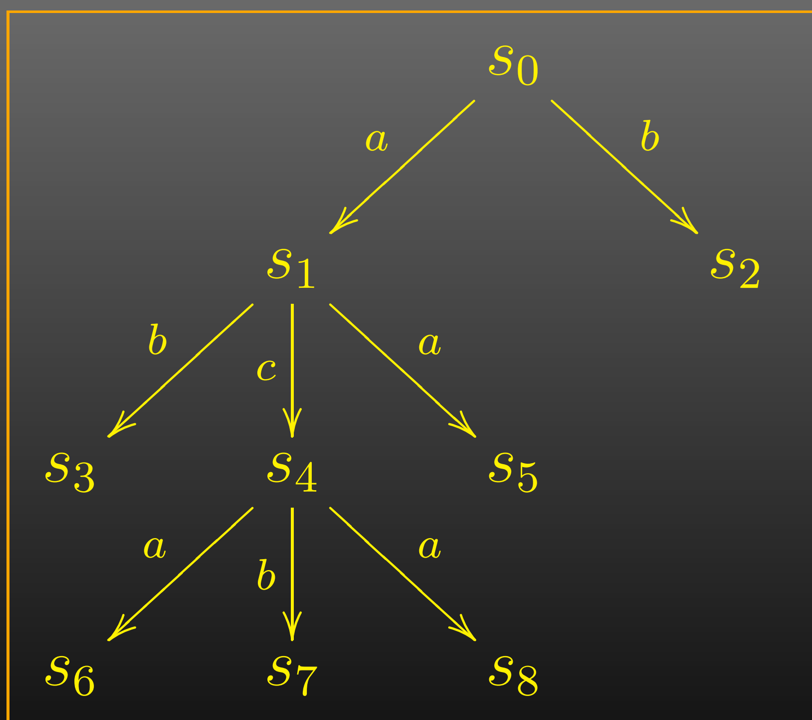
LTS

$\mathcal{P}(A \times \_)$



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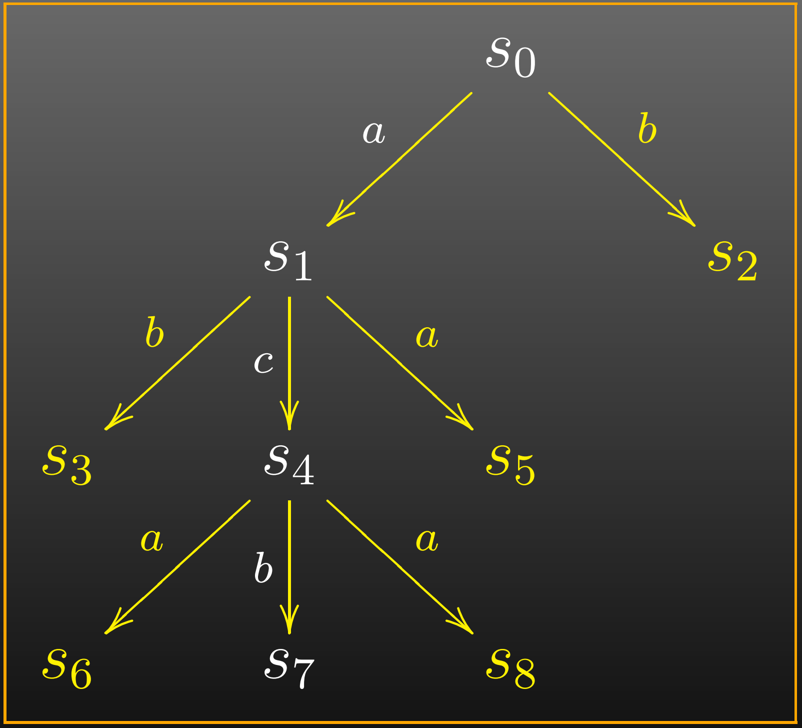


Execution (thin and fat):

$s_0 \xrightarrow{a} s_1 \xrightarrow{c} s_4 \xrightarrow{b} s_7$

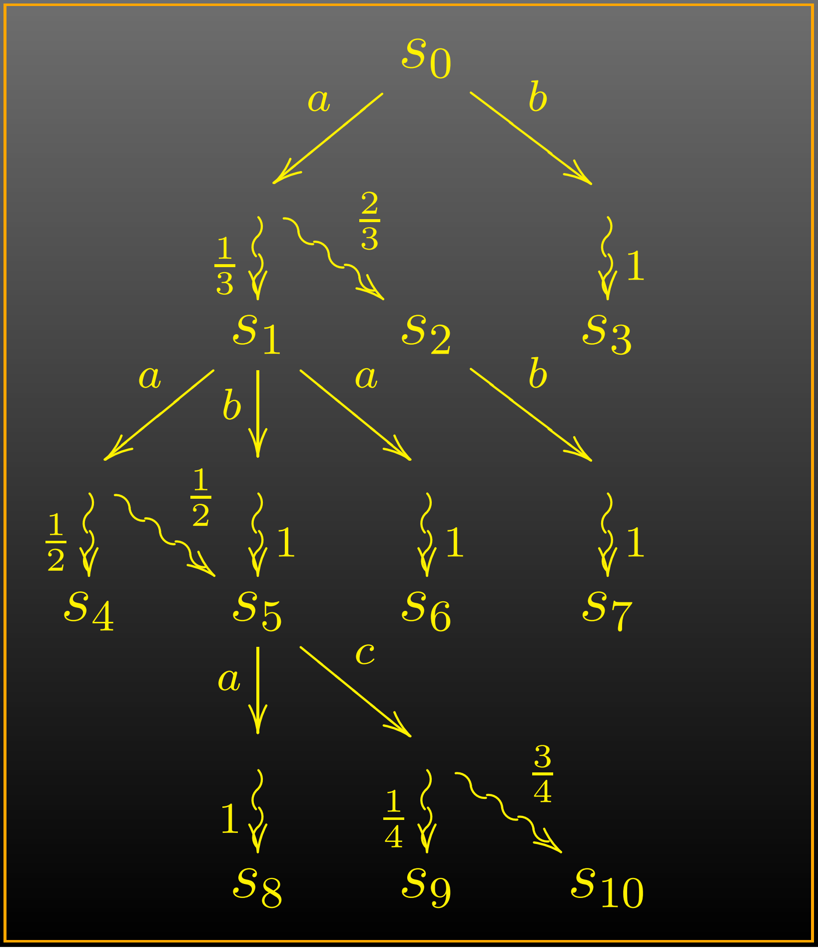
Trace (thin and fat): acb

Scheduler (deterministic):

$\xi : S \times (A \times S)^* \to A \times S + 1$

Simple Segala systems

$\mathcal{P}(A \times \mathcal{D})$



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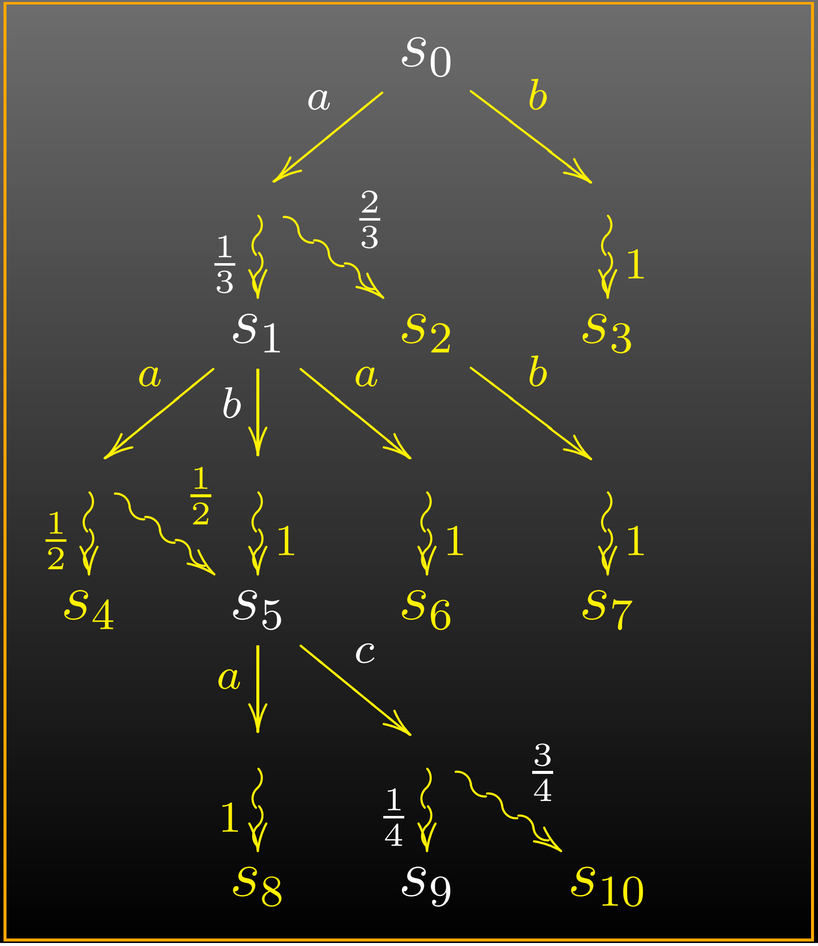

Execution (thin):

$$s_0 \xrightarrow{a,\mu} s_1 \xrightarrow{b} s_5 \xrightarrow{c,\nu} s_9$$

$$\mu = \left(s_1 \mapsto \tfrac{1}{3}, s_2 \mapsto \tfrac{2}{3}\right),$$

$$\nu = \left(s_9 \mapsto \tfrac{1}{4}, s_{10} \mapsto \tfrac{3}{4}\right)$$

Trace (fat): via schedulers

Scheduler (deterministic):

$$\xi : S \times (A \times \mathcal{D}(S) \times S)^* \to A \times \mathcal{D}(S) + 1$$

For $\mathcal{P}F$-coalgebras

* Execution?

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For $\mathcal{PF}$-coalgebras

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  - Hasuo&Jacobs, CALCO'05
    Hasuo&Jacobs&Sokolova, CMCS'06, LMCS'07
    Jacobs, CMCS'08

    (all on fat traces)

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* Scheduler?

  $\xi : S \times (F(S) \times S)^* \to F(S) + 1$

Coalgebraic fat traces

For $\mathcal{T}F$ - coalgebras, if ♣, then

$$\begin{array}{ccc} F_{\mathcal{K}\ell(\mathcal{T})}I & \qquad & F_{\mathcal{K}\ell(\mathcal{T})}I \\ \eta_I \circ \alpha \Big\downarrow \cong & \qquad & \eta_{FI} \circ \alpha^{-1} \Big\uparrow \cong \\ I & \qquad & I \end{array}$$

is initial   is final   in $\mathcal{K}\ell(\mathcal{T})$

[for $\alpha : FI \xrightarrow{\cong} I$ the initial F-algebra in $\mathbf{Sets}$]

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is initial   is final   in $\mathcal{K}\ell(\mathcal{T})$

[for $\alpha : FI \xrightarrow{\cong} I$ the initial F-algebra in $\mathbf{Sets}$]

♣ involves: existence of α, lifting of F to $\mathcal{K}\ell(\mathcal{T})$ via a distributive law, order-enriched $\mathcal{K}\ell(\mathcal{T})$

Coalgebraic fat traces ♣

For $X \xrightarrow{c} \mathcal{T}FX$ in $\mathbf{Sets}$ $\quad$ ($X \xrightarrow{c} F_{\mathcal{K}\ell(\mathcal{T})}X$ in $\mathcal{K}\ell(\mathcal{T})$)

Coalgebraic fat traces ♣

For $X \stackrel{c}{\to} \mathcal{T}FX$ in $\mathbf{Sets}$ ($X \stackrel{c}{\to} F_{\mathcal{K}\ell(\mathcal{T})}X$ in $\mathcal{K}\ell(\mathcal{T})$)

there exists a unique fat trace map $\mathrm{ftr}_c : X \to \mathcal{T}I$ in $\mathbf{Sets}$ by coinduction:

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in $\mathcal{K}\ell(\mathcal{T})$

$$\begin{array}{ccc}
F_{\mathcal{K}\ell(\mathcal{T})}X & \xrightarrow{F_{\mathcal{K}\ell(\mathcal{T})}(\mathrm{ftr}_c)} & F_{\mathcal{K}\ell(\mathcal{T})}I \\
{\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \alpha}\ \cong \\
X & \xrightarrow[\mathrm{ftr}_c]{} & I
\end{array}$$

Coalgebraic fat executions ♣

For $X \stackrel{c}{\longrightarrow} \mathcal{T}FX$ in $\mathbf{Sets}$

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For $X \xrightarrow{c} \mathcal{T}FX$ in $\mathbf{Sets}$

if $F(X \times \_)$ has an initial algebra $\alpha_X : F(X \times I_X) \xrightarrow{\cong} I_X$ then there exists a unique fat execution map

$$\mathrm{fexc}_c : X \to \mathcal{T}I_X \text{ in } \mathbf{Sets}$$

by coinduction:

Coalgebraic fat executions ♣

For $X \stackrel{c}{\to} \mathcal{T}FX$ in $\mathbf{Sets}$

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by coinduction:

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$$\begin{array}{ccc}
F(X \times X) & \xrightarrow{F_{\mathcal{K}\ell(\mathcal{T})}(id \times \mathrm{fexc}_c)} & F(X \times I_X) \\
{\scriptstyle F_{\mathcal{K}\ell(\mathcal{T})} J(\delta) \circ c} \uparrow & & \uparrow {\scriptstyle J\alpha_X^{-1}} \, \cong \\
X & \xrightarrow[\mathrm{fexc}_c]{} & I_X
\end{array}$$

From executions to traces

By initiality in **Sets**, we get a projection map:

$$
\begin{array}{ccc}
F(X \times I_X) & \stackrel{F(\mathrm{id} \times \pi_X)}{\dashrightarrow} & F(X \times I) \\
\alpha_X \Big\downarrow \cong & & \Big\downarrow \alpha \circ F(\pi_2) \\
I_X & \stackrel{\pi_X}{\dashrightarrow} & I
\end{array}
$$

From executions to traces

By initiality in $\mathbf{Sets}$, we get a projection map:

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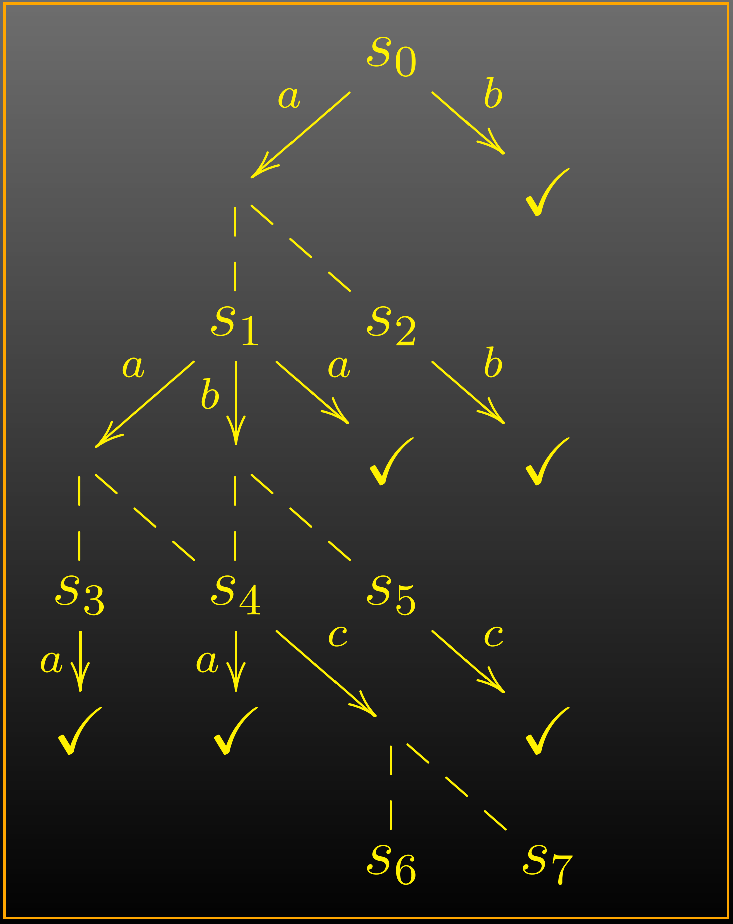
and the execution-to-trace equation is

$$\mathrm{ftr}_c \quad = \quad J(\pi_X) \circ \mathrm{fexc}_c \qquad \text{in } \mathcal{K}\ell(\mathcal{T})$$

Binary trees with output

$\mathcal{P}(A + A \times \_^2)$



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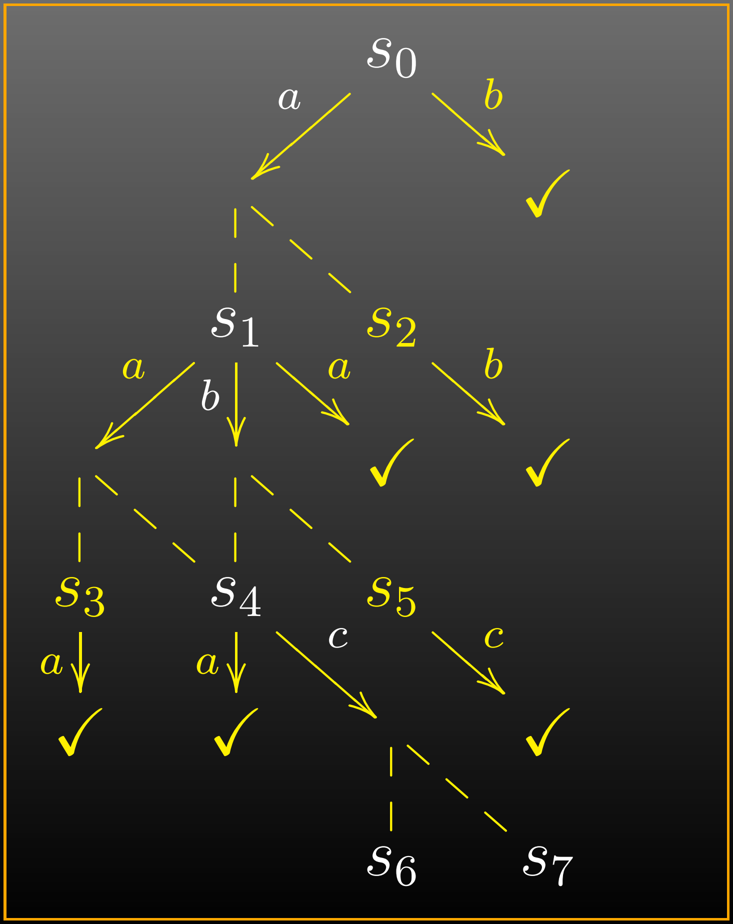

Execution (thin):

$$s_0 \xrightarrow{a,\langle s_1,s_2\rangle} s_1 \xrightarrow{b,\langle s_4,s_5\rangle} s_4 \xrightarrow{c,\langle s_6,s_7\rangle} s_6$$

not a fat execution!

Trace (fat): $\langle a, \langle b, c, c\rangle, b\rangle$

Scheduler (deterministic):

$$\xi : S \times (A \times S^2 \times S)^* \to (A + A \times S^2) + 1$$

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- what are thin executions and traces coalgebraically?

Splitting functors

Any subpower functor F [wpp, with $\rho : F \Rightarrow \mathcal{P}$, ...]

splits as $F_\emptyset(X) + F_\bullet(X) \xrightarrow{\cong} F(X)$

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* both $F_{\emptyset}$ and $F_{\bullet}$ are functors
* $F_{\emptyset}(X) = F(0)$
* $F_{\bullet}$ is subpower via $F_{\bullet} \Rightarrow F \Rightarrow \mathcal{P}$
* there is a natural map $\mathrm{split} : F \Rightarrow \mathcal{P}(F(0) + F_{\bullet} \times \mathit{id})$

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thin trace map by coinduction:

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$$\begin{array}{ccc} F(0) + F_\bullet(1) \times X & \xrightarrow{\mathit{id}+\mathit{id}\times \mathrm{ttr}_c} & F(0) + F_\bullet(1) \times L \\ c_{\mathrm{lt}} \uparrow & & \uparrow \cong \\ X & \xrightarrow{\mathrm{ttr}_c} & L \end{array}$$

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for the “thinned” coalgebra:

$$c_{\mathrm{lt}} = \mathcal{P}id + (F_{\bullet}(!) \times id) \circ \mu \circ \mathcal{P}\mathrm{split} \circ c$$

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From fat to thin traces one gets via a paths map ... difficult

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$$\mathrm{split} : A + A \times X^2 \to \mathcal{P}(A + A \times X^2 \times X)$$

$\mathrm{split}(a) = a$
$\mathrm{split}(\langle a, x_1, x_2 \rangle) = \{\langle \langle a, x_1, x_2 \rangle, x_1 \rangle, \langle \langle a, x_1, x_2 \rangle, x_2 \rangle\}$

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$$a \in c_{lt}(x) \iff a \in c(x)$$
$$\langle a, y \rangle \in c_{lt}(x) \iff \exists z. \langle a, y, z \rangle \in c(x) \vee \langle a, z, y \rangle \in c(x)$$

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and thin traces are as expected...

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as before, by:

- copying states
- changing the coalgebra structure using split
- coinduction

we get $\mathrm{texc}_c : X \to \mathcal{P}(L_X)$

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* thin executons and thin traces are related via a projection from L_X to L

Back to schedulers

F - subpower, $\Xi = X \times (F_\bullet X \times X)^*$ - “thin executions”

$$X \xrightarrow{\mathsf{in}} \Xi \underset{\mathsf{last}}{\overset{\mathsf{first}}{\rightrightarrows}} X$$

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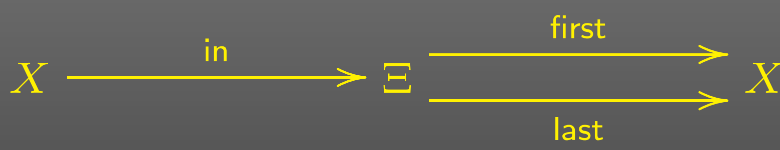
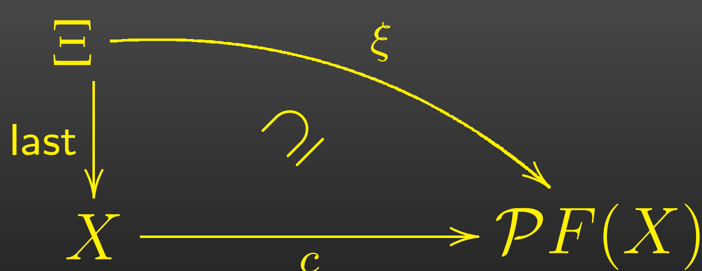


ξ is a non-deterministic scheduler for $c : X \to \mathcal{P}F(X)$ if



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ξ is a σ-type scheduler for $c : X \to \mathcal{P}F(X)$ if

$$\begin{array}{ccc} \Xi & \overset{\xi}{\dashrightarrow} & SF(X) \\ {\scriptstyle \mathsf{last}}\downarrow & \supseteq & \downarrow{\scriptstyle \sigma} \\ X & \underset{c}{\longrightarrow} & \mathcal{P}F(X) \end{array}$$

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deterministic, randomized, non-deterministic

Coalgebra under scheduler

ξ - a scheduler for $c \colon X \to \mathcal{P}F(X)$

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$$\begin{array}{ll}
\Xi & \xrightarrow{\langle \mathit{id}, \xi \rangle} \Xi \times SF(X) \\
 & \xrightarrow{\mathit{id} \times \sigma} \Xi \times \mathcal{P}F(X) \\
 & \xrightarrow{\mathsf{st}} \mathcal{P}(\Xi \times (F(0) + F_\bullet(X))) \\
 & \xrightarrow{\mathcal{P}(\mathsf{dist})} \mathcal{P}(\Xi \times F(0) + \Xi \times F_\bullet(X)) \\
 & \xrightarrow{\mathcal{P}(\pi_2 + \langle \mathit{id}, \pi_2 \rangle)} \mathcal{P}(F(0) + \Xi \times F_\bullet(X) \times F_\bullet(X)) \\
 & \xrightarrow{\mathcal{P}(\mathit{id} + \mathsf{st})} \mathcal{P}(F(0) + F_\bullet(\Xi \times F_\bullet(X) \times X)) \\
 & \xrightarrow{\mathcal{P}(\mathit{id} + F_\bullet(\mathsf{cons}))} \mathcal{P}F(\Xi) \qquad .
\end{array}$$

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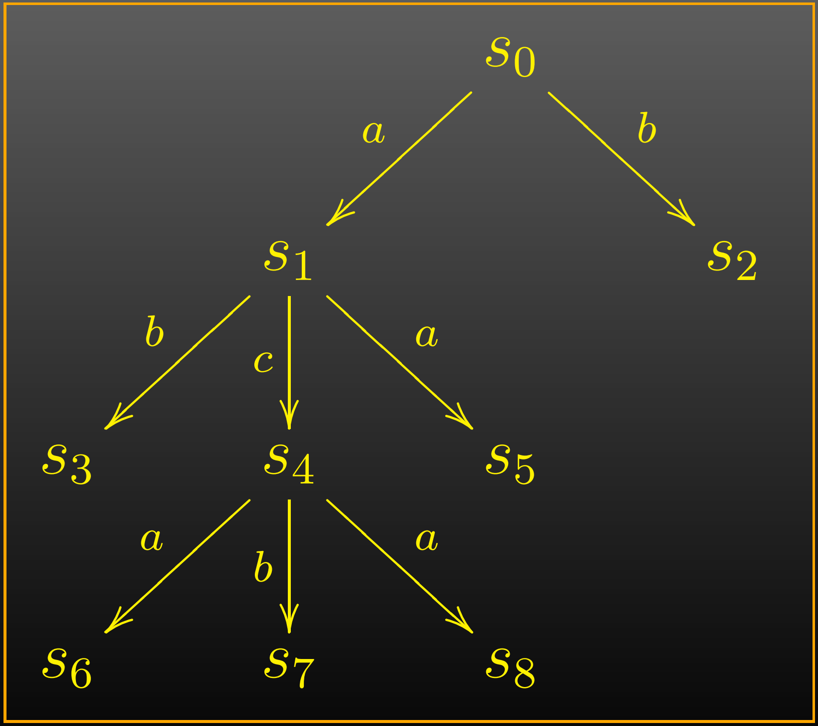

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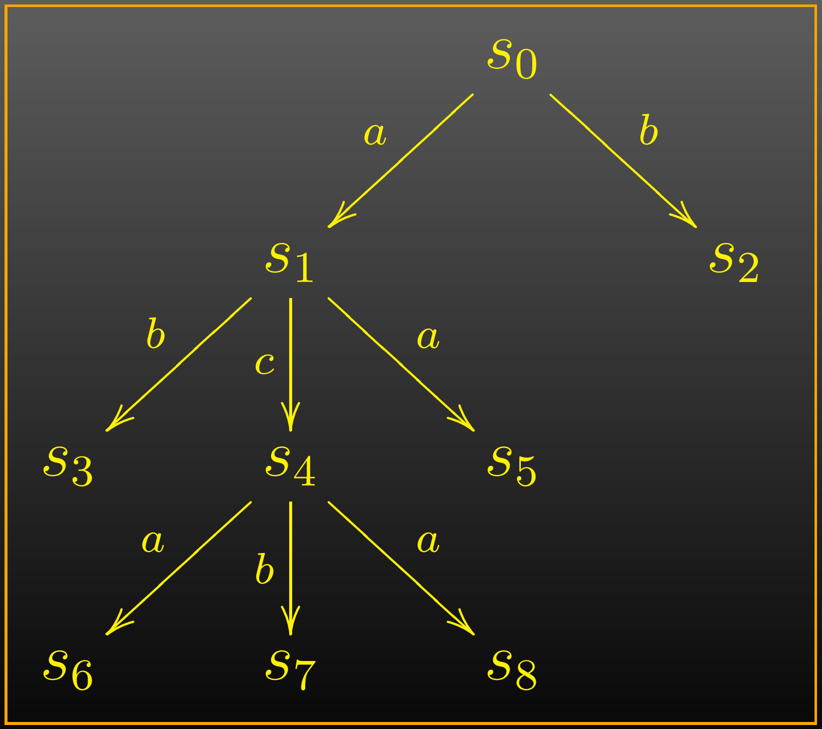

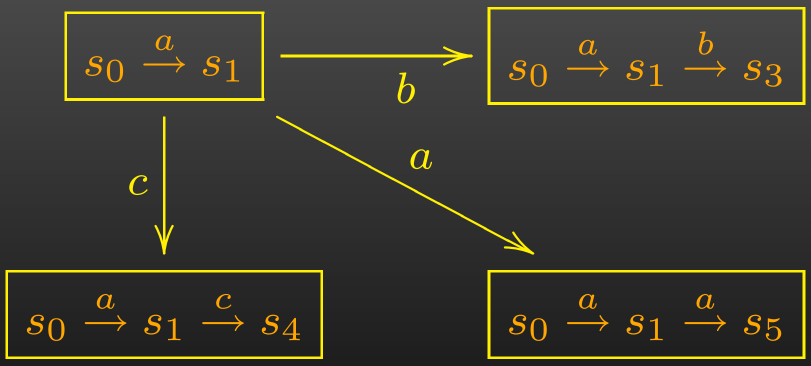

Scheduler traces

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Soundness: The scheduler traces of a coalgebra are contained in its traces.

Completeness?

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scheduled traces = (fat) traces

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Conjecture: $\sigma : S \Rightarrow \mathcal{P}$ is complete iff for any set X

$$\forall x \in X. \exists \alpha \in S. x \in \sigma(\alpha)$$

[anything can be scheduled]

Conclusions

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$X \to TF(X)$	**Thin** $\mathcal{P}$	**Fat** T
Traces	$F_\bullet(1)^\star \times F(0)$ $= \mu Y.\, F(0) + F_\bullet(1) \times Y$	$\mu Y.\, F(Y)$
Executions	$(F_\bullet(X) \times X)^\star \times F(0)$ $= \mu Y.\, F(0) + (F_\bullet(X) \times X) \times Y$	$\mu Y.\, F(X \times Y)$

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$X \to TF(X)$	**Thin** $\mathcal{P}$	**Fat** T
Traces	$F_\bullet(1)^\star \times F(0)$ $= \mu Y.\, F(0) + F_\bullet(1) \times Y$	$\mu Y.\, F(Y)$
Executions	$(F_\bullet(X) \times X)^\star \times F(0)$ $= \mu Y.\, F(0) + (F_\bullet(X) \times X) \times Y$	$\mu Y.\, F(X \times Y)$

* many open questions remain