Adequacy for Infinitary Algebraic Effects

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Monads for Computational Effects

- Moggi: Monads as notions of computation:
  $T(X)$ = computations for elements of X.
- Examples: exceptions, state, interactive I/O, (probabilistic) nondeterminism, continuations, and combinations thereof.
- Computational λ-Calculus:
  $\sigma \to \tau$ modelled as $[\![\sigma]\!] \Rightarrow T([\![\tau]\!])$.
- Algebraic Theory of Effects: emphasises the *effect constructors*, the operations that give rise to the effects (example: probabilistic choice).
- Monads and Operational Semantics: Relation starting point (work with John Power) for algebraic theory of effects.
- Treat infinitary case here, with a smoother presentation of earlier work.

Two examples of monads and operational semantics

Finite Nondeterminism

$$T(X) = \mathcal{F}^+(X) = \{u \mid u \subseteq_{\mathrm{fin}} X, u \neq \emptyset\}$$

$$M \,\mathtt{or}\, N \longrightarrow M$$

Probabilistic Computation

$$T(X) = \mathcal{D}_f(X) = \{\sum_{i=1}^{n} \lambda_i x_i \mid \lambda_i \geq 0, \sum \lambda_i = 1, x_i \in X\}$$

$$M + N \xrightarrow{1/2} M \qquad \text{(perhaps!)}$$

Two more examples of monads and operational semantics

Printing

$$T(X) = A^* \times X$$

$$\langle \texttt{print}_{\,a}; M, w\rangle \longrightarrow \langle M, aw\rangle$$

$$\texttt{print}_{\,a}; M \xrightarrow{a} M$$

Exceptions

$$T(X) = X + E$$

$$\texttt{raise}_{\,e} \downarrow_e$$

$$\frac{M \downarrow_{e_i}}{\texttt{try}\, x{:}\sigma \Leftarrow M \,\texttt{in}\, M' \,\texttt{unless}\, \{e_1 \Rightarrow M_1 \mid \cdots \mid e_n \Rightarrow M_n\} \longrightarrow M_i}$$

Effects are Achieved by Operations (Effect Constructors)

- Nullary

$$\texttt{raise}_e$$

- Unary

$$\texttt{print}_a; -$$

- Binary

$$- \texttt{or} - \qquad - + -$$

Question

How can we find a common ground for these operational semantics, say in the context of Moggi's computational λ-calculus with operations?

Outline

The computational λ-calculus

Signature Σ_{e} of operations $f : n$

Types

$$\sigma ::= \mathtt{b} \mid \mathtt{unit} \mid \sigma \times \sigma \mid \sigma \to \sigma$$

where $\mathtt{b}$ ranges over the *base types* $\mathtt{nat}$ and $\mathtt{bool}$

Terms

$$\begin{array}{lll} M & ::= & x \mid f(M_1, \ldots, M_n) \mid 0 \mid \mathtt{succ}\,(M) \mid \mathtt{prec}\,(M) \mid \mathtt{zero}\,(M) \mid \\ & & \underline{tt} \mid \underline{ff} \mid \mathtt{if}\; M \;\mathtt{then}\; M \;\mathtt{else}\; M \mid \\ & & * \mid (M, M) \mid \mathtt{fst}\; M \mid \mathtt{snd}\; M \mid \lambda x : \sigma . M \mid MM \end{array}$$

Typing Rules

$$\frac{\Gamma \vdash M_i : \sigma \quad (i = 1, n)}{\Gamma \vdash f(M_1, \ldots, M_n) : \sigma}$$

Denotational Semantics

Category **C** has finite products

Strong Monad T

Kleisli Exponentials

$$\mathbf{C}(z \times x, Ty) \cong \mathbf{C}(z, x \Rightarrow y)$$

Operators

$$f_x : (Tx)^n \to Tx \qquad (x \in \mathrm{Ob}(\mathbf{C}))$$

Natural Numbers $\quad \mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$

Truth Values $\quad \mathbb{T} \cong \mathbb{1} + \mathbb{1}$

Denotational Semantics (cntnd.)

Types

$$[\![\sigma]\!]$$

Terms

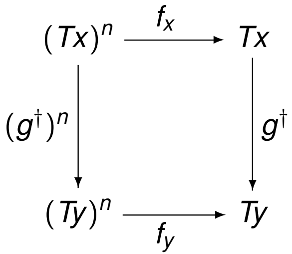
$$[\![\Gamma]\!] \xrightarrow{[\![M]\!]} T([\![\sigma]\!]) \qquad\qquad (\Gamma \vdash M : \sigma)$$

Example

$$[\![f(M_1, \ldots, M_n)]\!] = f_{[\![\sigma]\!]}\circ([\![M_1]\!], \ldots, [\![M_n]\!])$$

Algebraic Operations

Naturality



Remark Then $[\![E[f(M_1, \ldots, M_n)]]\!] = [\![f(E[M_1], \ldots, E[M_n])]\!]$

Example Finite Nondeterminism

$$TX = \mathcal{F}^+(X) \qquad g^\dagger(u) = \bigcup_{x \in u} g(x) \qquad X \,\mathtt{or}\, Y = X \cup Y$$

and so we have naturality: $g^\dagger(u \cup v) = g^\dagger(u) \cup g^\dagger(v)$

Outline

Operational Semantics

Reductions

$$\begin{cases} M \longrightarrow N & \\ M \xrightarrow{f_i} N & (1 \leq i \leq n) \\ M \downarrow_f & (f : 0) \end{cases}$$

Values

$$V ::= 0 \mid \mathtt{succ}\,(V) \mid V \mid \underline{tt} \mid \underline{ff} \mid * \mid (V, V) \mid \lambda x : \sigma.\, M$$

$$\mathrm{Val}_\sigma =_{\mathrm{def}} \{ V \mid \vdash V : \sigma \}$$

Evaluation Contexts

$$\begin{aligned} E \quad ::= \quad & [\,] \mid \mathtt{succ}\,(E) \mid \ldots \mid \mathtt{if}\, E \,\mathtt{then}\, M \,\mathtt{else}\, M \mid \\ & (E, M) \mid (V, E) \mid \mathtt{fst}\,(E) \mathtt{snd}\,(E) \mid EM \mid VE \end{aligned}$$

Operational Semantics (cntnd.)

Redexes R and their reductions:

$$\texttt{zero}\,(0) \longrightarrow \underline{tt} \qquad \texttt{if}\,\underline{tt}\,\texttt{then}\,M\,\texttt{else}\,N \longrightarrow M$$
$$\texttt{fst}\,((V, V')) \longrightarrow V \qquad (\lambda x:\sigma.\,M)V \longrightarrow M[V/x]$$

$$f(M_1, \ldots, M_n) \xrightarrow{f_i} M_i \qquad (i = 1, n)$$

Rules

$$\frac{R \longrightarrow N}{E[R] \longrightarrow E[N]} \qquad \frac{R \xrightarrow{f_i} N}{E[R] \xrightarrow{f_i} E[N]} \qquad E[f()] \downarrow_f$$

Fact Every closed term is either a value or else is analysable uniquely into one of the forms $E[R]$ or $E[f()]$.

Operational Semantics (cntnd.)

Big Step Semantics

$$
\begin{array}{lcl}
M \Rightarrow V & \equiv & M \longrightarrow^* V \\
M \stackrel{f_i}{\Longrightarrow} N & \equiv & M \longrightarrow^* \stackrel{f_i}{\longrightarrow} N \\
M \Downarrow_f & \equiv & M \longrightarrow^* N \downarrow_f
\end{array}
$$

Facts

1. $\longrightarrow$ is determinate
2. $\longrightarrow \cup \stackrel{f_i}{\longrightarrow}$ is well-founded

Effect Values and Evaluation

Effect Values $\quad t ::= V \mid f(t_1, \ldots, t_n)$

$T_{\Sigma_e}(\mathrm{Val}_\sigma)$ is the set of closed effect values of type σ.

Evaluation $\quad |\cdot| : \mathrm{Terms}_\sigma \longrightarrow \mathrm{Val}_\sigma \quad$ is the unique function st:

$$|M| = \begin{cases} V & (\text{if } M \Longrightarrow V) \\ f(|M_1|, \ldots, |M_n|) & (\text{if } M \overset{f_i}{\Longrightarrow} M_i) \\ f() & (\text{if } M \Downarrow_f) \end{cases}$$

Adequacy

Theorem

Adequacy: $[\![M]\!] = [\![\, | M | \,]\!]$

Proof.

Via an equational proof system, showing that $\vdash M = |M|$. □

Outline

Example: Probabilistic Computation

$$T(X) = \mathcal{D}_\omega(X) = \{\sum_{i=1}^{n} \lambda_i x_i\} \qquad \nu +_X \nu' = \frac{1}{2}\nu + \frac{1}{2}\nu'$$

For $M, V : \sigma$ set:

$$\mathrm{Prob}(M, V) = \sum\{2^{-|w|} \mid M \stackrel{w}{\Rightarrow} V, w \in \{+_1, +_2\}^*\}$$

Fact

$\mathrm{Prob}(-, V)$ is the unique $\theta : \mathrm{Terms}_\sigma \rightarrow [0, 1]$ st.:

$$\theta(M) = \begin{cases} 1 & (M \Longrightarrow V) \\ 0 & (M \Longrightarrow V' \neq V) \\ \frac{1}{2}\theta(M_1) + \frac{1}{2}\theta(M_2) & (M \stackrel{+_i}{\Longrightarrow} M_i) \end{cases}$$

Connection with General Operational Semantics

Fact

$$h(|M|) = \sum_V \mathrm{Prob}(M, V)\delta_V$$

where h is the unique homomorphism:

$$h\colon T_{\Sigma_{\mathrm{e}}}(\mathrm{Val}_\sigma) \longrightarrow \mathcal{D}_\omega(\mathrm{Val}_\sigma)$$

Proof.

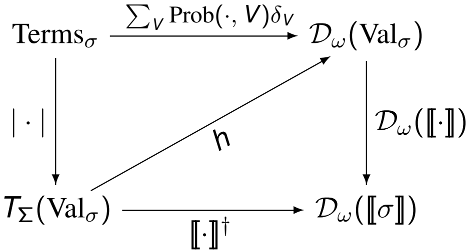
$h(|\cdot|)(V)$ obeys the equation for θ. □

Adequacy for Probabilistic Computation

Theorem

For any $M : \sigma$ we have: $[\![M]\!] = \sum_{V:\sigma} \mathrm{Prob}(M, V)[\![V]\!]$

Proof



Corollary For any $M : \mathtt{nat}$ we have: $[\![M]\!](m) = \mathrm{Prob}(M, \underline{m})$

Recursive Definitions

Syntax

$$\mathtt{Rec}\,(f\!:\!\sigma \to \tau, x\!:\!\sigma.\, M)$$

defines an $f\!:\!\sigma \to \tau$ such that $f(x) = M$

New Redex

$$\mathtt{Rec}\,(f\!:\!\sigma \to \tau, x\!:\!\sigma.\, M) \longrightarrow \lambda x\!:\!\sigma.\, M[\mathtt{Rec}\,(f\!:\!\sigma \to \tau, x\!:\!\sigma.\, M)/f]$$

Semantics

$$\begin{cases} C, T\ \omega\mathrm{Cpo}\text{-enriched;} \\ C_T\ \omega\mathrm{Cppo}\text{-enriched;} \\ \text{strength } \mathrm{st} \text{ strict.} \end{cases}$$

Interpretation of Recursive Definitions

Get

$$[\![\Gamma]\!] \times [\![\sigma \to \tau]\!] \times [\![\sigma]\!] \xrightarrow{[\![M]\!]} T[\![\tau]\!]$$

then

$$[\![\Gamma]\!] \times [\![\sigma \to \tau]\!] \xrightarrow{\mathrm{Curry}[\![M]\!]} [\![\sigma \to \tau]\!]$$

then

$$C([\![\Gamma]\!], [\![\sigma \to \tau]\!]) \longrightarrow C([\![\Gamma]\!], [\![\sigma \to \tau]\!])$$

and then take a least fixed-point.

Evaluation

- $|M|$ can be infinite
- Example:

$$|\texttt{Rec}\,(f, x : \texttt{unit}\,.\,\texttt{print}\,_a(f(x))(*)| = \texttt{print}\,_a(\texttt{print}\,_a(\ldots))$$

- Terms of infinite depth.

$$\begin{array}{rcl} \mathrm{CT}_{\Sigma_e}(X) & = & \text{finite and } \infty \text{ partial } \Sigma_e\text{-trees over } X \\ & = & \text{free continuous } \Sigma_e\text{-algebra over } X \text{ in } \omega\text{-}\mathbf{Cpo} \end{array}$$

- Finite partial terms

$$t ::= x \mid f(t_1, \ldots, t_n) \mid \Omega$$

Evaluation Function

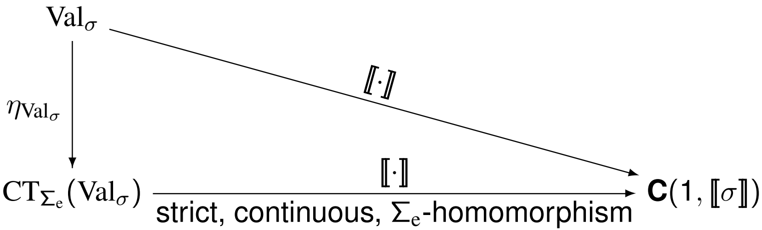
$$|\cdot|_\sigma : \mathrm{Terms}_\sigma \longrightarrow \mathrm{CT}_{\Sigma_e}(\mathrm{Val}_\sigma)$$

is the unique function such that:

$$|M|_\sigma = \begin{cases} V & (M \Rightarrow V) \\ f(|M|_1, \ldots, |M|_n) & (M \overset{f_i}{\Rightarrow} M_i) \\ f() & (M \Downarrow_f) \\ \Omega & (\text{otherwise}) \end{cases}$$

Outline

Semantics of $\mathrm{CT}_{\Sigma_\mathrm{e}}(\mathrm{Val}_\sigma)$



Adequacy Theorem (cntnd.)

Theorem

(Adequacy) For any $M:\sigma$, $[\![M]\!] = [\![|M|]\!]$

Mezei-Wright like result, but for a higher-order language.

Proof.

One direction, $\geq$, using soundness of relevant equational theory. Other direction by introducing a language with approximants $\mathrm{Rec}_n(f:\sigma \rightarrow \tau, x:\sigma.\, M)$. Don't know a proof using logical relations. □

Example: Probabilistic Computation

$$\mathbf{C} = \omega\text{-}\mathbf{Cpo}$$

$$\begin{array}{rcl} T(P) & = & \mathcal{V}(P) \\ & = & \text{Evaluations on } P \\ & = & \{\mu : \mathcal{O}(P) \to [0,1] \mid \mu \text{ is modular, continuous and strict}\} \end{array}$$

$$\nu +_P \nu' = \frac{1}{2}\nu + \frac{1}{2}\nu' \qquad \Omega_P = (V \mapsto 0)$$

Example: Probabilistic Computation (cntnd.)

For $M, V : \sigma$ set: $\mathrm{Prob}(M, V) = \sum\{2^{-|w|} \mid M \stackrel{w}{\Rightarrow} V\}$
as before (but this sum can now be countably infinite)

Fact

$\mathrm{Prob}(-, V)$ *is the unique* $\theta : \mathrm{Terms}_\sigma \to [0, 1]$ *st.:*

$$\theta(M) = \begin{cases} 1 & (M \Longrightarrow V) \\ 0 & (M \Longrightarrow V' \neq V) \\ \frac{1}{2}\theta(M_1) + \frac{1}{2}\theta(M_2) & (M \stackrel{+_i}{\Longrightarrow} M_i) \\ 0 & (\mathit{otherwise}) \end{cases}$$

(Uniqueness by the Banach fixed-point theorem.)

Connection with General Operational Semantics

Let

$$h : \mathrm{CT}_{\Sigma_e}(\mathrm{Val}_\sigma) \rightarrow \mathcal{V}(\mathrm{Val}_\sigma)$$

be the unique strict cts. Σ-homomorphism extending η (= δ)

Fact

$$h(|M|) = \sum_V \mathrm{Prob}(M, V)\delta_V$$

Proof.

$h(|\cdot|)(V)$ obeys the equation for θ. □

Adequacy for Probabilistic Computation

Theorem

(Adequacy for Probabilistic Computation with Recursion) For any $M : \sigma$:

$$[\![M]\!] = \sum_{V:\sigma} \mathrm{Prob}(M, V)[\![V]\!]$$

Proof.

As before! □

Corollary

For any $M : \mathtt{nat}$: $\quad \mathrm{Prob}(M, \underline{m}) = [\![M]\!](m)$

Outline

Example: Interactive I/O

- Working in **Set**, take initial solution of:

$$TX \cong (TX)^I + (O \times TX) + X$$

$$\mathrm{input}_X : TX^I \to TX \qquad \mathrm{output}_X : O \times TX \to TX$$

- Working in ω-**Cpo**$_\perp$, take initial solution of:

$$TP \cong ((TP)^I + (O \times TP) + P)_\perp$$

$$\mathrm{input}_P : TP^I \to TP \qquad \mathrm{output}_P : O \times TP \to TP \qquad \Omega_P : TP$$

General Operations

Examples of effect constructors

$$+_X : TX^2 \longrightarrow TX \qquad \mathrm{raise} : E \longrightarrow TX$$
$$\mathrm{read} : TX^I \longrightarrow TX \qquad \mathrm{write} : O \times TX \longrightarrow TX$$

General Form

$$\mathrm{op}_X : O \times TX^I \longrightarrow TX$$

Sum of product form

$$O = \prod_{k=1}^{p} O_k \qquad I = \sum_{i=1}^{m} \prod_{j=1}^{n_i} I_{ij}$$

Get

$$\mathrm{op}_X : \prod_{k=1}^{p} O_k \times \prod_{i=1}^{n} TX^{\prod_{j=1}^{n_i} I_{ij}} \longrightarrow TX$$

Syntax

A Multisorted First Order Signature

$$\Sigma_{\mathrm{b}} = \begin{cases} \text{Basic Types} & b & \text{(including } \texttt{bool}, \texttt{nat}\text{)} \\ \text{Function Symbols} & g : \mathbf{b} \to b' & \text{(including } \mathrm{succ} \text{ etc)} \\ \text{Relation Symbols} & R : \mathbf{b} & \text{(including } \texttt{zero}\text{)} \end{cases}$$

Arity Types Ranged over by a, a subset of the Basic Types

Operation Signature Σ_{e} consists of operation symbols

$$\mathrm{op} : \mathbf{b}; \mathbf{a}_1, \ldots, \mathbf{a}_n$$

Syntax, cntnd.

Term Language

$$M ::= g(\mathbf{M}) \mid R(\mathbf{M}) \mid \mathrm{op}_{\mathbf{M}}((\mathbf{x}_1 : \mathbf{a}_1).\, M_1, \ldots, (\mathbf{x}_n : \mathbf{a}_n).\, M_n)$$

Typing Rule

$$\frac{\Gamma \vdash \mathbf{M} : \mathbf{b} \qquad \Gamma, \mathbf{x}_i : \mathbf{a}_i \vdash M_i : \sigma \quad (i = 1, n)}{\Gamma \vdash \mathrm{op}_{\mathbf{M}}((\mathbf{x}_1 : \mathbf{a}_1).\, M_1, \ldots, (\mathbf{x}_n : \mathbf{a}_n).\, M_n) : \sigma} \qquad (\text{for } \mathrm{op} : \mathbf{b}; \mathbf{a}_1, \ldots, \mathbf{a}_n)$$

Semantics

- We take **C** to be **Set** (or ω-**Cpo**). I am unsure how to proceed more generally.
- Each arity basic type *a* is interpreted by a countable set (or by a countable discrete ω-cpo)
- We take a fixed first-order interpretation of Σ_{b} (and denote it using $[\![\cdot]\!]$)
- Each operation $\mathrm{op}:\mathbf{b};\mathbf{a}_1,\ldots,\mathbf{a}_n$ is interpreted by a family of maps:

$$\mathrm{op}_x = [\![\mathbf{b}]\!] \times (Tx)^{[\![\mathbf{a}_1]\!]} \times \ldots \times (Tx)^{[\![\mathbf{a}_n]\!]} \longrightarrow Tx$$

subject to an evident naturality requirement.

Outline

Operational Semantics

Values

$$V ::= \underline{v}$$

for any b and $v \in [\![b]\!]$

Evaluation Contexts

$$E ::= g(\mathbf{V}E\mathbf{M}) \mid R(\mathbf{V}E\mathbf{M}) \mid \mathrm{op}_{\mathbf{V}E\mathbf{M}}((\mathbf{x}_1 : \mathbf{b}).\, M_1, \ldots, (\mathbf{x}_n : \mathbf{b}_n).\, M_n)$$

Redexes

$$\begin{array}{llll} g(\underline{\mathbf{v}}) & \longrightarrow & \underline{v'} & (\text{if } [\![g]\!](\mathbf{v}) = v') \\ R(\underline{\mathbf{v}}) & \longrightarrow & \underline{tt} & (\text{if } [\![R]\!](\mathbf{v}) = tt) \\ R(\underline{\mathbf{v}}) & \longrightarrow & \underline{ff} & (\text{if } [\![R]\!](\mathbf{v}) = ff) \end{array}$$

Operational Semantics (cntnd.)

New Relation

$$M \xrightarrow{\mathrm{op}!\underline{\mathbf{v}}?i,\underline{\mathbf{w}}} N$$

for $\mathrm{op}:\mathbf{b};\mathbf{a}_1,\ldots,\mathbf{a}_n$, $\mathbf{v}\in[\![\mathbf{b}]\!]$, $1\leq i\leq n$ and $\mathbf{w}\in[\![\mathbf{a}_i]\!]$

New Redex

$$\mathrm{op}_{\underline{\mathbf{v}}}((\mathbf{x}_1:\mathbf{a}_1).\,M_1,\ldots,(\mathbf{x}_n:\mathbf{a}_n).\,M_n) \xrightarrow{\mathrm{op}!\underline{\mathbf{v}}?i,\underline{\mathbf{w}}} M_i[\mathbf{w}/\mathbf{x}_i]$$

Outline

Adequacy: Set

New Effect Values

$$\mathrm{op}_{\underline{\mathbf{v}}}(\langle t_{\mathbf{w}_1}\rangle_{\mathbf{w}_1 \in [\![\mathbf{a}_1]\!]}, \ldots, \langle t_{\mathbf{w}_n}\rangle_{\mathbf{w}_n \in [\![\mathbf{a}_n]\!]}) \qquad (\mathbf{v} \in [\![\mathbf{b}]\!])$$

These are infinitary.

Evaluation Function

$$|M| \quad = \quad \mathrm{op}_{\underline{\mathbf{v}}}(\langle |M_{\mathbf{w}_1}|\rangle_{\mathbf{w}_1 \in [\![\mathbf{a}_1]\!]}, \ldots, \langle |M_{\mathbf{w}_n}|\rangle_{\mathbf{w}_n \in [\![\mathbf{a}_n]\!]}) \quad (M \xrightarrow{\mathrm{op}!\underline{\mathbf{v}}?i,\underline{\mathbf{w}}} M_{\mathbf{w}_i})$$

Adequacy

$$[\![M]\!] = [\![|M|]\!]$$

Adequacy ω-**Cpo**

- Effect Values The ω-cpo CT_Σ where Σ consists of $[\![\mathbf{a}_1]\!] \times \ldots \times [\![\mathbf{a}_n]\!]$-ary function symbols $\mathrm{op}_{\underline{\mathbf{v}}}$ (for $\mathbf{v} \in [\![\mathbf{b}]\!]$).
- Instead of finite elements one uses the well-founded infinitary ones:

$$t ::= V \mid \mathrm{op}_{\mathbf{v}}(\langle t_{\mathbf{w}_1} \rangle_{\mathbf{w}_1 \in [\![\mathbf{a}_1]\!]}, \ldots, \langle t_{\mathbf{w}_n} \rangle_{\mathbf{w}_n \in [\![\mathbf{a}_n]\!]}) \mid \Omega$$

- Get $|M| \in \mathrm{CT}_\Sigma$, and then adequacy, with statement as before.

I/O Example

Input

$$\mathrm{input}_P : \mathbb{1} \times TP^I \longrightarrow TP$$

Ignore trivial ‘output type’ $\mathbb{1}$ and get:

$$\mathrm{input}((x : \mathrm{in}).\, M) \xrightarrow{\mathrm{input}?\underline{w}} M[\underline{w}/x]$$

Output

$$\mathrm{output}_P : O \times TP^{\mathbb{1}} \longrightarrow TP$$

Ignore trivial ‘input type’, and get:

$$\mathrm{output}_{\underline{v}}(M) \xrightarrow{\mathrm{output}!\underline{v}} M$$

Adequacy for I/O

One reads off for $\vdash M : \sigma$:

$$
\begin{array}{lllllll}
(i) & [\![M]\!] & = & \bot & \text{iff} & M \Uparrow \\
(ii) & [\![M]\!] & = & x \in [\![\sigma]\!] & \text{iff} & \exists V.\, M \Rightarrow V \\
 & & & & & \wedge \quad [\![V]\!] = x \\
(iii) & [\![M]\!] & = & \mathrm{input}(f) & \text{iff} & \forall w \in I.\, \exists N.\, M \xrightarrow{\mathrm{input}?w} N \\
 & & & & & \wedge \quad [\![N]\!] = f(w) \\
(iv) & [\![M]\!] & = & \mathrm{output}_v(u) & \text{iff} & M \xrightarrow{\mathrm{output}!v} N \\
 & & & & & \wedge \quad [\![N]\!] = u
\end{array}
$$

Behavioural Equality (Example: Probabilistic Choice)

Set The following are equivalent for closed M:`bool`

$$[\![M]\!] = [\![N]\!]$$

$$\mathrm{Prob}(\mathrm{M}, \underline{\mathrm{tt}}) = \mathrm{Prob}(\mathrm{N}, \underline{\mathrm{tt}}) \quad \wedge \quad \mathrm{Prob}(\mathrm{M}, \underline{\mathrm{ff}}) = \mathrm{Prob}(\mathrm{N}, \underline{\mathrm{ff}})$$

ω-**Cpo** The following are equivalent for closed M:`bool`

$$[\![M]\!] \leq [\![N]\!]$$

$$\mathrm{Prob}(\mathrm{M}, \underline{\mathrm{tt}}) \leq \mathrm{Prob}(\mathrm{N}, \underline{\mathrm{tt}}) \quad \wedge \quad \mathrm{Prob}(\mathrm{M}, \underline{\mathrm{ff}}) \leq \mathrm{Prob}(\mathrm{N}, \underline{\mathrm{ff}})$$

Contextual Equivalence

Set For $M, N : \sigma$ set

$$M \simeq N \quad \text{iff} \quad \forall C[\cdot].[\![C[M]]\!] = [\![C[N]]\!]$$

where $C[\cdot]$ ranges over contexts of type `bool` such that $C[M], C[N]$ are closed terms of type `bool`.

We have: $[\![M]\!] = [\![N]\!] \quad \Rightarrow \quad M \simeq N$

ω-**Cpo** For $M, N : \sigma$ set

$$M \precsim N \quad \text{iff} \quad \forall C[\cdot].[\![C[M]]\!] \leq [\![C[N]]\!]$$

We have:

$$[\![M]\!] \leq [\![N]\!] \quad \Rightarrow \quad M \precsim N$$

Applicative Simulation (ω-**Cpo**)

Let $\preceq^e$ be the relation on closed terms defined as above, but restricted to the following 'extensional' contexts:

$$C^e ::= [\cdot] \mid C^e V \mid \texttt{fst}\ C^e \mid \texttt{snd}\ C^e$$

where V is closed.[1] Extend this to open terms by substitution.

Question Do we have:

$$\frac{M \preceq^e N}{M \preceq N}$$

Should probably restrict to ω-**Cpo**-monads given by an inequational theory with a least element.

[1] changed to values from general terms after the talk

Deconstructors

- Operations are *effect constructors*. There are also *effect deconstructors*, e.g., Benton and Kennedy's, more generally:

$$\frac{M : \sigma \qquad H : \mathtt{exc} \to \tau \qquad x : \sigma \vdash N : \tau}{\mathtt{try}\, M\, \mathtt{with}\, H\, \mathtt{as}\, x : \sigma\, \mathtt{in}\, N : \tau}$$

- Plotkin and Pretnar (ESOP '09) gave a general version of this (in a call-by-push-value context)
- Can we get a general adequacy theorem with both effect constructors and deconstructors present?

State

- The monad, in the case of **Set**, is $TX = (S \times X)^S$ where $S = \mathrm{Val}^L$.
- The operations are

$$\mathrm{update}: (L \times \mathrm{Val}) \times TX \to TX \qquad \mathrm{lookup}_X : L \times TX^{\mathrm{Val}} \longrightarrow TX$$

with some equations determining $T(X)$ as a free algebra.

- Get $\mathrm{lookup}_{\underline{l}}((x : \mathtt{val}).\, M) \xrightarrow{\mathrm{lookup}!\underline{l}?\underline{v}} M[\underline{v}/x]$
- Prefer $\langle s, \mathrm{lookup}_{\underline{l}}((x : \mathtt{val}).\, M)\rangle \longrightarrow \langle s, M[\underline{s(l)}/x]\rangle$
- Power and Plotkin (MFPS '08) characterise $S \times X$ as a tensor $S \otimes T(X)$ of a coalgebra and an algebra. This could be the beginning of a general theory.

Some Other Questions

- What to do around (pre)sheaves/fresh names?
- How to treat process calculi? The rules do not come out as above; concurrency is a binary deconstructor (Plotkin and Pretnar, van Glabeek and Plotkin, to appear).
- Relation to co-algebraic treatments of operational semantics: bisimulation, formats, modal logics?
- So...

Finally

Thank you for your attention

and:

Happy Birthday Peter!