Institut für Theoretische Informatik

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Complete Iterativity for Algebras with Effects

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Motivation

Algebras with effects

$a : HA \longrightarrow A$ algebra

Motivation

Algebras with effects

$a : HA \to A + \{\bot\}$ partial algebra

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$a : HA \to A + \{\bot\}$ partial algebra

$a : HA \to \mathcal{P}A$ nondet. algebra

$\dots$

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Completely iterative algebras (cias)

$a : HA \to A$ algebra where every equation system

$$\begin{array}{l} x \approx \sigma(x, y) \\ y \approx a_0 \\ \dots \end{array}$$

has a unique solution

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$a : HA \to \mathcal{P}A$ nondet. algebra

...

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Questions

- What are the right notions of equation systems and solutions?
- How can cias with certain effects be characterised?

Approach

- Lift analytic functors H canonically to $\mathbf{Set}_M$

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- Notions of cias with effects
  - Kleisli-cias
  - λ-cias

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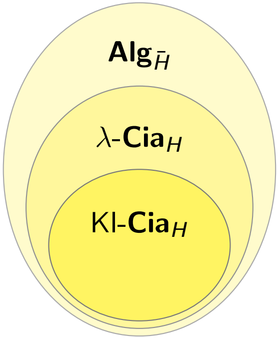

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- Lift analytic functors H canonically to $\mathbf{Set}_M$
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  - Kleisli-cias
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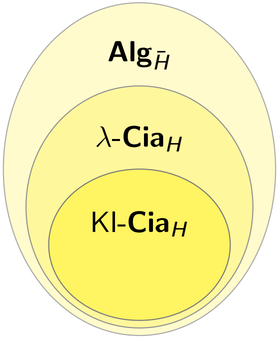
- Free Kleisli-/λ-cias from free H-algebras



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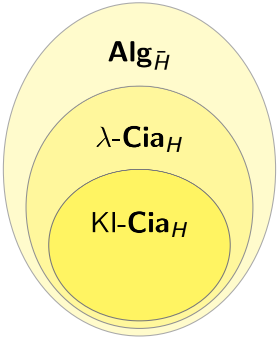

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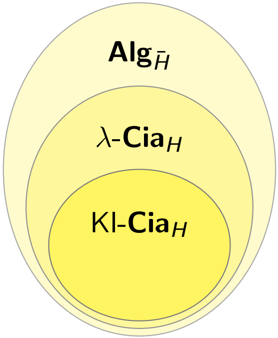

Based on I. Hasuo, B. Jacobs, A. Sokolova 2007

Outline

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Effects as Monads

- Well-known technique: E. Moggi (late 1980s)
- Observation: $A + \{\bot\}$, $\mathcal{P}A$, ... are of the form MA for some set monad (M, η, μ)

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Monad constituent	**Meaning**
M	type of effect
$\eta : \mathrm{Id} \to M$	computations without effects
$\mu : MM \to M$	composition of effects
axioms	the "properties one wants"

Example Effects

Effect	**Monad**
partiality	maybe monad $MA = A + \{\bot\}$
nondeterminism	powerset monad $MA = \mathcal{P}A$

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probabilistic nondeterminism	subdistribution monad $MA = \{d : A \to [0,1] \mid \sum_{a \in A} d(a) \leq 1\}$
and many more	...

Kleisli Category

Definition

Kleisli category $\mathbf{Set}_M$ of M:

- objects: all sets
- morphisms from A to B: all maps $f : A \to MB$

Remark

Inclusion functor $J : \mathbf{Set} \to \mathbf{Set}_M$ with $JA = A$ and $Jf = \eta \cdot f$

Liftings to $\mathbf{Set}_M$

Definition

$\bar{H}$ is a lifting of H to $\mathbf{Set}_M$ if $\bar{H}J = JH$.

Definition

Distributive law of H over M: natural transformation $\lambda : HM \to MH$ + two axioms

Proposition (P. S. Mulry 1994)

Liftings of H to $\mathbf{Set}_M$ $\iff$ distributive laws $HM \to MH$.

Canonical Liftings

Definition

Analytic functor: coproduct of functors $(-)^n/G$

Examples

1. Polynomial functors
2. Unordered pairs functor
3. Bag functor

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Theorem

Any analytic functor has a canonical lifting to the Kleisli category of a commutative monad.

- This extends a previous result of I. Hasuo, B. Jacobs and A. Sokolova.

Commutative Monads

Examples

- Maybe monad $MA = A + \{\bot\}$
- Powerset monad $MA = \mathcal{P}A$
- Environment monad $MA = A^E$
- Subdistribution monad
  $MA = \{d : A \to [0,1] \mid \sum_{a \in A} d(a) \leq 1\}$

Non-example

- List monad $MA = \coprod_{n \in \mathbb{N}} A^n$

Canonical Distributive Laws

Examples

Polynomial functor H_Σ and

- maybe monad: $\lambda_A : H_\Sigma(A + \{\bot\}) \to H_\Sigma A + \{\bot\}$
  identity on $H_\Sigma A$, otherwise constant to $\bot$
- powerset monad: $\lambda_A : H_\Sigma \mathcal{P} A \to \mathcal{P} H_\Sigma A$
  cartesian product

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Cias (without effects)

Definitions (S. Milius 2005)

H functor on a category $\mathcal{A}$ with finite coproducts.

- flat equation morphism (in A): $e : X \to HX + A$

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- solution (of e in $a : HA \to A$): $e^\dagger : X \to A$ such that

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\ HX + A & \xrightarrow[He^\dagger + A]{} & HA + A \end{array}$$

  commutes

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Example

$\tau : T \to HT$ final H-coalgebra $\iff$ τ^{-1} initial cia

Adding Effects: Kleisli-/λ-Cias

Definition

Kleisli-cia: cia for a lifting $\bar{H}$

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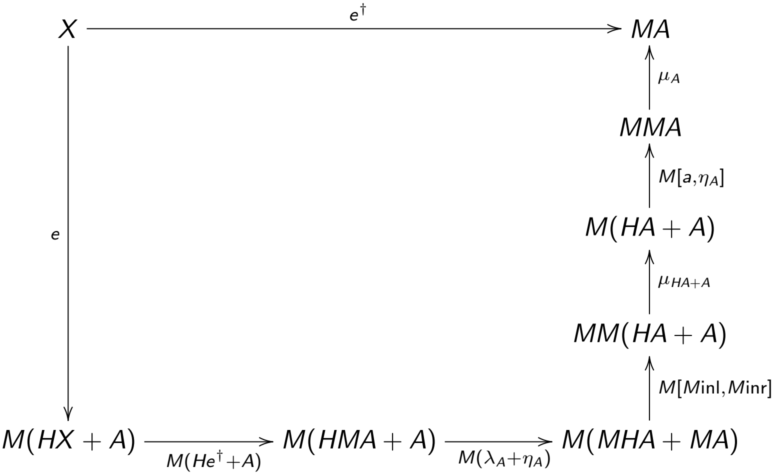

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Definitions

H set functor with lifting $\bar{H}$ to $\mathbf{Set}_M$

- M-equation morphism (in A): $e : X \to HX + MA$

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- completely λ-iterative algebra (λ-cia): $a : HA \to MA$ such that for every $e : X \to HX + MA$ there exists a unique solution

A Concrete λ-Cia

Example

- $M = \mathcal{P}$, $\bar{H}_\Sigma$ canonical lifting of H_Σ to $\mathbf{Set}_M$
- $F_\Sigma Y$ finite Σ-trees on Y

A Concrete λ-Cia

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- Variables are solved uniquely to finite tree unfoldings or $\emptyset$

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- $\Sigma = \{*\}$

$$x_1 \approx x_2 * x_3 \qquad x_2 \approx \left\{ \begin{array}{c} * \\ / \quad \backslash \\ y_1 \quad y_2 \end{array}, y_3 \right\} \qquad x_3 \approx \left\{ \begin{array}{c} * \\ / \quad \backslash \\ y_3 \quad y_4 \end{array} \right\}$$

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$$e^\dagger(x_1) = \left\{ \begin{array}{c} * \\ / \qquad \backslash \\ \begin{array}{c} * \\ / \ \backslash \\ y_1 \ y_2 \end{array} \quad \begin{array}{c} * \\ / \ \backslash \\ y_3 \ y_4 \end{array} \end{array}, \ \begin{array}{c} * \\ / \qquad \backslash \\ y_3 \quad \begin{array}{c} * \\ / \ \backslash \\ y_3 \ y_4 \end{array} \end{array} \right\}$$

Kleisli-Cias and λ-Cias

	Algebra a	**Equations** e	**Solutions** $e^\dagger$
cias in **Set**	$HA \to A$	$X \to HX + A$	$X \to A$
Kleisli-cias	$HA \to MA$	$X \to M(HX + A)$	$X \to MA$
λ-cias	$HA \to MA$	$X \to HX + MA$	$X \to MA$

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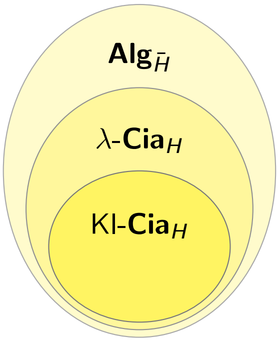

Proposition

Every Kleisli-cia is a λ-cia.

Outline

Free λ-Cias

Assumptions

- H set functor with lifting $\bar{H}$ to $\mathbf{Set}_M$
- $\mathbf{Set}_M$ cpo-enriched with left-strict composition
- $\bar{H}$ locally monotone

Theorem

The free H-algebra ϕ_Y on Y yields the free $\bar{H}$-algebra, Kleisli-cia and λ-cia $J\phi_Y$ on Y.

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The free H-algebra ϕ_Y on Y yields the free $\bar{H}$-algebra, Kleisli-cia and λ-cia $J\phi_Y$ on Y.

Example

Canonical lifting of H_Σ to $\mathbf{Set}_M$ for maybe, powerset, subdistribution monad: the free λ-/Kleisli-cia on Y is carried by all finite Σ-trees on Y.

Free λ-Cias (ctd.)

Theorem

The free H-algebra ϕ_Y on Y yields the free $\bar{H}$-algebra, Kleisli-cia and λ-cia $J\phi_Y$ on Y.

- Theorem is based on results of I. Hasuo, B. Jacobs and A. Sokolova.
- It does not apply to the environment monad, but:

Proposition

Canonical lifting of H to $\mathbf{Set}_M$ for environment monad: the final $H + Y$-coalgebra $\tau_Y : TY \to HTY + Y$ yields the free λ-/Kleisli-cia $J(\tau_Y^{-1} \cdot \mathsf{inl})$ on Y.

Characterisations of λ-Cias

Theorem

For the maybe monad and the canonical lifting of a polynomial functor H_Σ the following are equivalent for $a : H_\Sigma A \to MA$:

1. *a is a λ-cia.*
2. *a is a Kleisli-cia.*
3. *a is an $\bar{H}_\Sigma$-algebra "increasing" for some well-founded order.*

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A similar theorem holds for the powerset monad.

Example (for $H_\Sigma = (-)^2$, $M = \mathcal{P}$)

$(\mathbb{N} \setminus \{0,1\}, *)$ with $n * m = \{n \cdot m\}$ is a λ-/Kleisli-cia.

$x \approx y * z$	$\{8, 10, 12, 15\}$
$y \approx \{2,3\}$	$\{2,3\}$
$z \approx \{4,5\}$	$\{4,5\}$

$x \approx x * y$ $y \approx \{8\}$	$\emptyset$ $\{8\}$
$x \approx \{2, x*x\}$	$\{2^n \mid n \geq 1\}$

Characterisations of λ-Cias (ctd.)

Theorem

For the environment monad and the canonical lifting the following are equivalent for $a : HA \to MA$:

1. *a is a λ-cia.*
2. *a is a Kleisli-cia.*
3. *a is an $\bar{H}$-algebra such that $\pi_i \cdot a$ is a cia in* **Set** *for all $i \in E$.*

- The reason: the solution diagrams for λ-/Kleisli-cias decompose to solution diagrams in **Set**.

Outline

Conclusion

Summary

- Canonical liftings for analytic functors/commutative monads
- Notions of cias with effects
  - Kleisli-cias
  - λ-cias
- Results on cias with effects
  - Free Kleisli-/λ-cias for cpo-enriched $\mathbf{Set}_M$
  - Characterisation of Kleisli-/λ-cias for maybe, powerset and environment monad

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Open Questions

- When do we have λ-cias $=$ Kleisli-cias? Has this to do with commutative monads or is there a counterexample?
- Characterisation Theorems for analytic functors?
- Capture effects by Lawvere theories?

Thank you. . .

. . . for your attention!

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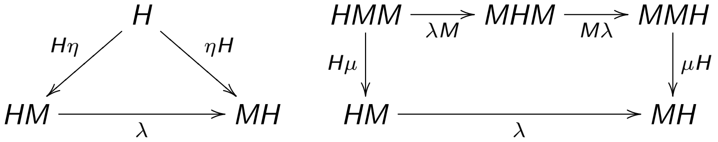
Liftings are Distributive Laws

Definition

$\bar{H}$ is a lifting of H to $\mathbf{Set}_M$ if $\bar{H}J = JH$.

Definition

Distributive law of H over M: natural transformation
$\lambda : HM \to MH$ such that



commute. →back

Proposition (P. S. Mulry 1994)

Liftings of H to $\mathbf{Set}_M$ $\iff$ distributive laws $HM \to MH$.

Existence of Further Liftings

In general, there exist other than the canonical liftings of H to $\mathbf{Set}_M$.

Examples

$H = \mathrm{Id}$, monad endomorphisms $\lambda : M \to M \iff$ distributive laws

1. Environment monad $M = (-)^E$ with $|E| = 2$:
   $\lambda = \mathrm{id} : (-)^2 \to (-)^2$ is canonical, but also
   $\lambda' = \mathsf{c} : (-)^2 \to (-)^2$ (the symmetry isomorphism) is a monad endomorphism.
2. Output monad $M = (-) \times O$ for some monoid O: any monoid endomorphism $h : O \to O$ extends to a monad morphism $\lambda = \mathrm{id} \times h : (-) \times O \to (-) \times O$. For the monoid $(\mathbb{N}, +, 0)$ there are infinitely many monoid endomorphisms: consider multiplication with any fixed natural number.

→back

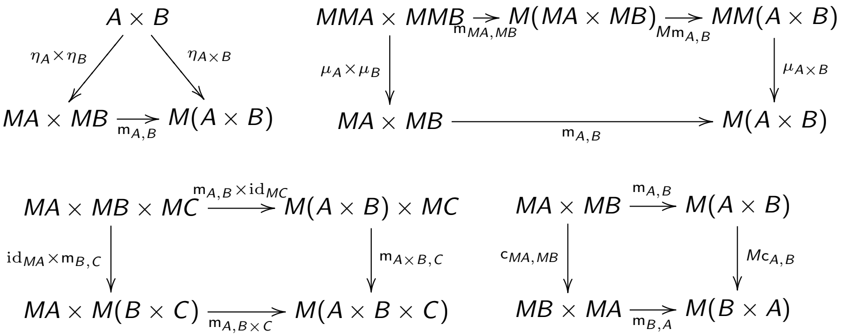
Commutative Monads

Commutative monads $\iff$ symmetric monoidal monads (A. Kock 1970/72)

$\longrightarrow$back

Definition

Symmetric monoidal monad (on **Set**): (M, η, μ) together with $\mathsf{m}_{A,B} : MA \times MB \to M(A \times B)$ natural in A and B such that the following diagrams commute:

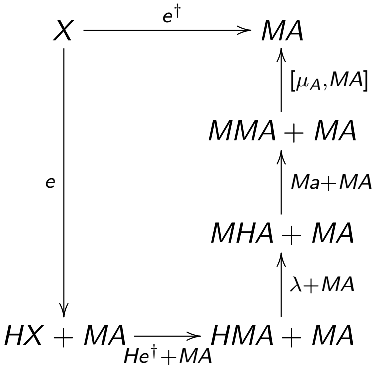


λ-Cias

Definitions

H set functor with lifting $\bar{H}$ to $\mathbf{Set}_M$

- M-equation morphism (in A): $e : X \to HX + MA$
- solution (of e in $a : HA \to MA$): $e^\dagger : X \to MA$ such that the right-hand diagram commutes in **Set**



- completely λ-iterative algebra (λ-cia): $a : HA \to MA$ such that for every $e : X \to HX + MA$ there exists a unique solution

Solution Preserving Morphisms

Definition

For any flat equation morphism $e : X \to HX + A$ in $\mathcal{A}$ and any morphism $f : A \to B$ define $f \bullet e = (HX + f) \cdot e$. A morphism f between cias A and B is called solution preserving if $(f \bullet e)^\dagger = f \cdot e^\dagger$ for every flat equation morphism e.

Proposition

For a morphism $f : A \to B$ between cias the following are equivalent:

1. *f is an H-algebra homomorphism.*
2. *f preserves solutions.*

Solution Preserving Morphisms (ctd.)

Definition

For any M-equation morphism $e : X \to HX + MA$ in A and any morphism $f : A \to MB$ define $f \bullet e = (HX + \mu_B \cdot Mf) \cdot e$. A morphism f between λ-cias A and B is called solution preserving if $(f \bullet e)^\dagger = \mu_B \cdot Mf \cdot e^\dagger$ for every flat equation morphism e.

Proposition

For a morphism $f : A \to MB$ between λ-cias the following are equivalent:

1. *f is an $\bar{H}$-algebra homomorphism.*
2. *f preserves solutions.*

Proper Subcategories

There exist λ-cias that are no Kleisli-cias.

Example

$H = \mathrm{Id}$, M list monad, $\lambda = \mathrm{id}$. Then

$$a : \{0,1\} \to M\{0,1\} \quad \text{with } a(0) = [1] \text{ and } a(1) = [1,1]$$

is a (unary) λ-cia: $\mu_{\{0,1\}} \cdot Ma$ has the unique fixed point [] and is increasing for the (well-founded) length/lexicographic order. But it is no Kleisli-cia since $x \approx [x, 1]$ has no solution in finite lists.

There exist $\bar{H}$-algebras that are no λ-cias.

Example

$H = \mathrm{Id}$, M powerset monad, $\lambda = \mathrm{id}$. The (unary) λ-cias are precisely the well-founded graphs; but clearly there also exist non-well-founded graphs.

→back

Cpo-Enriched Kleisli Categories

Definitions

- $\mathcal{A}$ is called cpo-enriched if each hom-set carries a cpo such that composition preserves joins of ω-chains.
- Composition of morphisms in $\mathcal{A}$ is called left-strict if for each morphism f the map $- \cdot f$ preserves the least element.
- An endofunctor H on $\mathcal{A}$ is called locally monotone if each derived function $\mathcal{A}(A, B) \to \mathcal{A}(HA, HB)$ is monotone.

Examples

1. Maybe monad $M = (-) + \{\perp\}$: MB carries flat cpo
2. Powerset monad $M = \mathcal{P}$: MB carries the inclusion cpo
3. Subdistribution monad $M = \mathbb{D}$: MB carries the pointwise cpo

In all examples, the cpos on MB pointwise induce cpos on each hom-set $\mathbf{Set}(A, MB)$. Furthermore, composition of morphisms is left-strict and each canonical lifting is locally monotone. →back

Unary λ-Cias

$H = \mathrm{Id}$ lifts to $\mathbf{Set}_M$ for any monad via $\lambda = \mathrm{id} : M \to M$.

Proposition

For $\lambda = \mathrm{id}$ the following are equivalent for $a : A \to MA$:

1. *a is a λ-cia.*
2. *$\mu_A \cdot Ma : MA \to MA$ has a unique fixed point $a_0 \in MA$ and is "increasing" for some well-founded order on $MA \setminus \{a_0\}$.*

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Examples

1. $\eta_A : A \to MA$ is a λ-cia iff $MA \cong 1$.
2. Maybe monad: $a_0 = \bot$; a is a λ-cia iff for some well-founded order $>$, $a(b) = \bot$ or $a(b) > b$ for all $b \in A$.
3. Powerset monad: $a_0 = \emptyset$; a is a λ-cia iff the dual of the corresponding graph is well-founded.