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Conway Games, coalgebraically

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Games and Logic

- Aristotle: writings on syllogisms intertwined with studies on use and aim of debate
- in medieval times Logic is called dialectics
- Buridan's Sophismata *e.g.* Nihil et Chimera suntne fratres?
- Brouwer: Mathematics should not degenerate into a game
- Tarski's definition of truth and Hintikka's infinite game
- Game Theoretic Semantics: Abelard $\forall$ and Eloise $\exists$
- Given any first-order sentence ϕ, interpreted in a fixed structure $\mathcal{A}$, player $\exists$ has a, deterministic, winning strategy for Hintikka's game G(ϕ) if and only if ϕ is true in $\mathcal{A}$ in the sense of Tarski.
- GTS can be extended to cover Kripke semantics in the case of modal logics, and generalizations, including Hennessy Milner logic.
- game theoretic account of bisimulation

Logic and Games

- Lorenzen and dialogue logic
- Proof theoretic semantics and dialogue games
- game theoretic denotational semantics

Our starting point: an effort to understand the game paradigms, arising in different contexts, starting from the tradition more directly connected to “ordinary games”.

Classical combinatorial games

- 2-player games, Left (L) and Right (R)
- games have positions
- L and R move in turn
- perfect knowledge: all positions are public to both players
- in any position there are rules which restrict L to move to any of certain positions (Left positions), while R may similarly move only to certain positions (Right positions)
- the game ends when one of the two players does not have any option

Many Games played on boards are combinatorial games: Nim, Domineering, Go, Chess.

Impartial games: for every position both players have the same set of moves.
Partizan games: L and R may have different sets of moves.

Moves, positions, plays

The players play by choosing elements of a set Ω, called the domain of moves of the game. As they choose, they build up an alternating sequence

$$\omega_1,\ \omega_2,\ \omega_3, \ldots$$

of elements of Ω. Infinite alternating sequences of moves are called plays. Finite sequences of elements of moves are called positions; they record where a play might have got to by a certain time.

Conway Games

- In the 1960s, Berlekamp, Conway, Guy introduced the theory of partizan games, firstly exposed in Conway's book "On Numbers and Games" .
- However, Conway focuses only on finite, i.e. terminating games. Infinite games are neglected as ill-formed or trivial, not interesting for "busy men".
- Some infinite (or loopy) games have been considered later, but focus on specific games or on some well-behaved classes of games. In any case games are fixed, *i.e.* infinite plays are all winning for either for L or for R players. No draws.

Infinite Games in Computer Science

Modern computing systems such as

- operating systems
- communication protocols
- controllers

are non-terminating reactive systems, i.e. systems interacting with their environment by exchanging information with it.

Infinite games are a fruitful metaphor for non-terminating reactive systems, they allow to capture in a natural way the perpetual interaction between system and environment.

Conway Games, formally

Games are identified with initial positions.
Any position p is determined by its Left and Right options, $p = (P^L, P^R)$.

The set $\mathcal{G}$ of games is inductively defined by:

- the empty game $(\{\}, \{\}) \in \mathcal{G}$;
- if $P, P' \subseteq \mathcal{G}$, then $(P, P') \in \mathcal{G}$.

Equivalently, $\mathcal{G}$ is the carrier of the initial algebra $(\mathcal{G}, id)$ of the functor $F : \text{Class}^* \to \text{Class}^*$, $F(X) = \mathcal{P}(X) \times \mathcal{P}(X)$.

Some simple games:

- $0 = (\{\}, \{\})$
- $1 = (\{0\}, \{\})$
- $-1 = (\{\}, \{0\})$
- $* = (\{0\}, \{0\})$

Winning Strategies

- A winning strategy for L player tells, at each step, reachable position, which is the next L move,*i.e.* L option, in response to any possible last move of R, *i.e.* R option,
- A winning strategy for R player tells, at each step, which is the next R move, *i.e.* R option, in response to any possible last move, *i.e.* L option, of L.
- A winning strategy for I player tells, at each step, which is the next move of the I player (the player who has started the game), in response to any possible last move of the II player.
- A winning strategy for II player tells, at each step, which is the next move of the II player (the player who has not started the game), in response to any possible last move of the I player.

Winning strategies are formalized as partial functions from positions to moves.

Winning Strategies on Simple Games

- On $0 = (\{\}, \{\})$, the I player will lose (independently whether he plays L or R), since there are no options. Thus the II player has a winning strategy.
- On $1 = (\{0\}, \{\})$ there is a winning strategy for L, since, if L plays first, then L has a move to 0, and R has no further move; otherwise, if R plays first, then he loses, since he has no moves.
- $-1 = (\{\}, \{0\})$ has a winning strategy for R.
- $* = (\{0\}, \{0\})$ has a winning strategy for the I player, since he has a move to 0, which is losing for the next player.

Conway Characterization Result on Games

Determinacy Theorem. Any game has a winning strategy either for L or for R or for I or for II.

Definition. Let $x = (X^L, X^R)$, $y = (Y^L, Y^R)$ be games.

$$x \gtrsim y \text{ iff } \forall x^R \in X^R. (y \not\gtrsim x^R) \wedge \forall y^L \in Y^L. (y^L \not\gtrsim x) .$$

– $x > y$ iff $x \gtrsim y \wedge y \not\gtrsim x$
– $x \sim y$ iff $x \gtrsim y \wedge y \gtrsim x$
– $x || y$ (x fuzzy y) iff $x \not\gtrsim y \wedge y \not\gtrsim x$

Characterization Theorem. Let x be a game. Then

$x > 0$	(x is positive)	iff	x has a winning strategy for L.
$x < 0$	(x is negative)	iff	x has a winning strategy for R.
$x \sim 0$	(x is zero)	iff	x has a winning strategy for II.
$x \|\| 0$	(x is fuzzy)	iff	x has a winning strategy for I.

Hypergames and non-losing Strategies

The set of Hypergames $\mathcal{H}$ is the carrier of the final coalgebra $(\mathcal{H}, id)$ of the functor $F : \text{Class}^* \to \text{Class}^*$ on classes of non-wellfounded sets, $F(X) = \mathcal{P}(X) \times \mathcal{P}(X)$.

Coinduction Principle. Two hypergames p, q are equal iff there exists a relation $\mathcal{R}$ s.t. $p\mathcal{R}q$, where $\mathcal{R}$ is a hyperbisimulation, i.e.

$$x\mathcal{R}y \implies (\forall x^L \in X^L.\exists y^L \in Y^L.x^L\mathcal{R}y^L) \wedge (\forall x^R \in X^R.\exists y^R \in Y^R.x^R\mathcal{R}y^R) .$$

Plays on hypergames can be non-terminating.
A non-terminating play is a draw.
The notion of winning strategy is replaced by that of non-losing strategy.

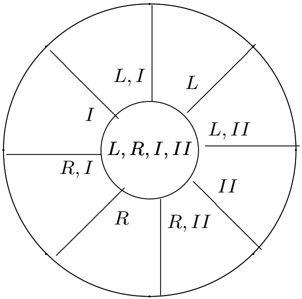
Simple hypergames

- $c = (\{c\}, \{c\})$. Any player (L, R, I, II) has a non-losing strategy, since there is only the non-terminating play consisting of infinite c's.
- $a = (\{b\}, \{\})$ and $b = (\{\}, \{a\})$. If L plays as II on a, then he immediately wins since R has no move. If L plays as I, then he moves to b, then R moves to a and so on, an infinite play is generated. This is a draw. Hence L has a non-losing strategy on a. Simmetrically, b has a non-losing strategy for R.

The Space of Hypergames

Theorem. Any hypergame has a non-losing strategy at least for one of the players L, R, I, II.

The space of hypergames:



Extending the relation $\gtrsim$ on hypergames

Problem: a direct extension of $\gtrsim$ on hypergames is not possible, since the associated operator is not monotonic.

Idea: define both relations $\gtrsim$ and $\not\gtrsim$ at the same time, as the greatest fixpoint of the monotone operator
$\Phi : \mathcal{P}(\mathcal{H} \times \mathcal{H}) \times \mathcal{P}(\mathcal{H} \times \mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H} \times \mathcal{H}) \times \mathcal{P}(\mathcal{H} \times \mathcal{H})$

$$\begin{aligned}\Phi(\mathcal{R}_1, \mathcal{R}_2) = (\{(x, y) \mid \forall x^R . y \mathcal{R}_2 x^R \ \wedge \ \forall y^L . y^L \mathcal{R}_2 x\}, \\ \{(x, y) \mid \exists x^R . y \mathcal{R}_1 x^R \ \vee \ \exists y^L . y^L \mathcal{R}_1 x\})\end{aligned}$$

Coinduction Principles: We call Φ-bisimulation a pair of relations $(\mathcal{R}_1, \mathcal{R}_2)$ such that $(\mathcal{R}_1, \mathcal{R}_2) \subseteq \Phi(\mathcal{R}_1, \mathcal{R}_2)$. The following principles hold:

$$\frac{(\mathcal{R}_1, \mathcal{R}_2) \ \Phi\text{-bisimulation} \quad x \mathcal{R}_1 y}{x \gtrsim y}$$

$$\frac{(\mathcal{R}_1, \mathcal{R}_2) \ \Phi\text{-bisimulation} \quad x \mathcal{R}_2 y}{x \not\gtrsim y}$$

Characterization Theorem on Hypergames

Theorem.

$x > 0$	(x is positive)	iff	x has a non-losing strategy for L.
$x < 0$	(x is negative)	iff	x has a non-losing strategy for R.
$x \sim 0$	(x is zero)	iff	x has a non-losing strategy for II.
$x \| 0$	(x is fuzzy)	iff	x has a non-losing strategy for I.

Remark. The relations $\gtrsim$ and $\not\gtrsim$ are not disjoint.
E.g. the game $c = (\{c\}, \{c\})$ is such that both $c \gtrsim 0$ and $c \not\gtrsim 0$ (and also $0 \gtrsim c$ and $0 \not\gtrsim c$) hold.
This is consistent with the fact that some hypergames have non-losing strategies for more than one player.

Combining Hypergames: coalgebraic sum

Using sum, a compound game can be built where, at each step, players can play on one of the components.

$$x + y = (\{x^L + y \mid x^L \in X^L\} \cup \{x + y^L \mid y^L \in Y^L\},\\ \{x^R + y \mid x^R \in X^R\} \cup \{x + y^R \mid y^R \in Y^R\})\ .$$

Hypergame Sum is given by the the final morphism $+ : (\mathcal{H} \times \mathcal{H}, \alpha_+) \longrightarrow (\mathcal{H}, \mathrm{id})$, where the coalgebra morphism $\alpha_+ : \mathcal{H} \times \mathcal{H} \longrightarrow F(\mathcal{H} \times \mathcal{H})$ is defined by

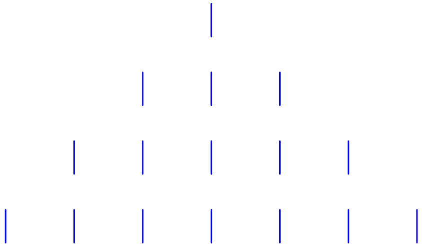
$$\alpha_+(x, y) = (\{(x^L, y) \mid x^L \in X^L\} \cup \{(x, y^L) \mid y^L \in Y^L\},\\ \{(x^R, y) \mid x^R \in X^R\} \cup \{(x, y^R) \mid y^R \in Y^R\})\ .$$

Hypergame sum resembles that of shuffling on processes. It coincides with interleaving, when impartial games are considered.

The Theory of Impartial Games: Nim

- Nim is played with a number of heaps of matchsticks.
- The legal move is to strictly decrease the number of matchsticks in any heap.
- A player unable to move because no sticks remain is the loser.

"Last year in Marienbad" configuration:



Nim as a Conway Game

The Nim game with one heap of size n can be represented as the Conway game $*n$, defined (inductively) by

$$*n = \{*0, *1, \ldots, *(n-1)\} \; .$$

Namely, with a heap of size n, the options of the next player consist in moving to a heap of size $0, 1, \ldots, n-1$.

Nim games correspond to von Neumann finite numerals in Set Theory.

Winning strategy: if $n = 0$, the II player wins; otherwise player I has a winning strategy, moving to $*0$.

General Nim with heaps of sizes $n_1, \ldots, n_k$: is the sum of k single-heap Nim games.

Sum of Nim numbers: $*n_1 + *n_2 = *n$.
The Nim sum amounts to binary sum without carries.
E.g. $*1 + *3 = *2$, since $01 \oplus 11 = 10$.

Grundy-Sprague Result on Impartial Games

Theorem. [Grundy39-Sprague35] Any impartial game behaves as a single-heap Nim game.

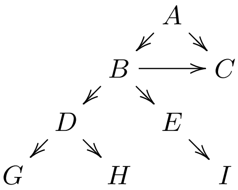
Mex Algorithm to compute (inductively) the Nim number:
If the Nim numbers of the options of x are $n_0, n_1, \ldots$, then the Nim number of x is the minimal excludent (mex) of $n_0, n_1, \ldots$
The mex of a list of numbers $n_0, n_1, \ldots$ is the least natural number which does not appear among $n_0, n_1, \ldots$

Graphs of Impartial Games

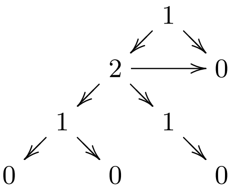
Impartial games correspond to wellfounded sets and can be represented as (acyclic) directed graphs

nodes: positions
directed edges: from a position p to a position q, when there is a move from p to q.

Game Graph:



Mex Marking:



Impartial Hypergames

Impartial hypergames can be represented by cyclic graphs.

Canonical hypergames extend Nim games with:

$$*\infty_\emptyset = \{*\infty_\emptyset\}$$

$$*\infty_K = \{*\infty_\emptyset\} \cup \{*k \mid k \in K\}$$

Lemma. $*\infty_K$ is winning for I iff $0 \in K$, otherwise it is a draw.

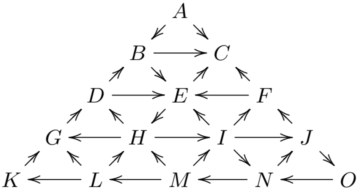
Theorem. Any impartial hypergame behaves either like a Nim game or like a hypergame of the shape $*\infty_K$.

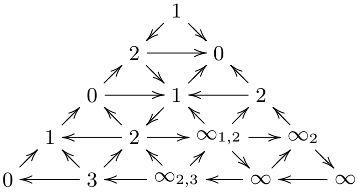
Hypergame Marking Algorithm, based on Smith Algorithm

A position p in the graph will be marked with the number n if the following conditions hold.

- Firstly, n must be the mex of all numbers that already appear as marks of any of the options of p.
  Secondly, each of the positions immediately following p which has not been marked with some number less than n must already have an option marked by n.
- We continue in this way until it is impossible to mark any further node with any ordinal number, and then attach the symbol ∞ to any remaining node.
- Finally, the label of a position marked as n is n, while the label of an unmarked position is the symbol ∞ followed by the labels of all marked options as subscripts.

Traffic Jam


A
B C
D E F
G H I J
K L M N O


1
2 0
0 1 2
1 2 $\infty_{1,2}$ ∞_2
0 3 $\infty_{2,3}$ ∞ ∞

Traffic Jams and Generalized Sums

- More than one vehicle is considered.
- Each town is big enough to accommodate all vehicles at once, if needed.
- At each step, the current player chooses a vehicle to move.

Such game corresponds to the sum of the hypergames with single vehicles.

To compute non-losing strategies, we use the generalized Nim sum, which amounts to the Nim sum extended to ∞-nodes as follows:

$$*n + *\infty_K = *\infty_K + *n = *\infty_{\{*k+*n \mid k \in K\}} \qquad *\infty_K + *\infty_H = *\infty\ .$$

Example: if vehicles are at positions H and I, then the game is winning for I player, since $*2 + *\infty_{1,2} = *\infty_{*2+*1,*2+*2} = *\infty_{3,0}$. While a game with vehicles in I and J is a draw, since $*\infty_{1,2} + *\infty_2 = *\infty$.

Future Work

- Alternative winning strategies. In the literature, various notions of winning strategies have been considered. For example, *misère* is the variant where the roles of winner and loser are exchanged. Moreover, various notions of winning strategies, especially devised for infinite games, have been considered.
- Compound hypergames. The (disjunctive) sum is used for building compound (hyper)games. However, there are several different ways of combining (hyper)games.
- Trace categories of hypergames and strategies. Joyal77 introduced a traced category of Conway games and winning strategies. It would be interesting to investigate analogous categories for hypergames. Cfr. also game categories of Abramsky er al., Hyland-Ong.