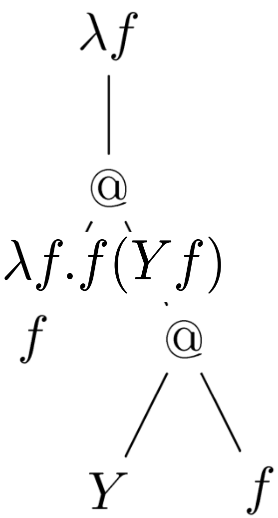
Semantics of Higher-Order Recursion Schemes

Jiří Adámek
Stefan Milius
Jiří Velebil

mail@stefan-milius.eu

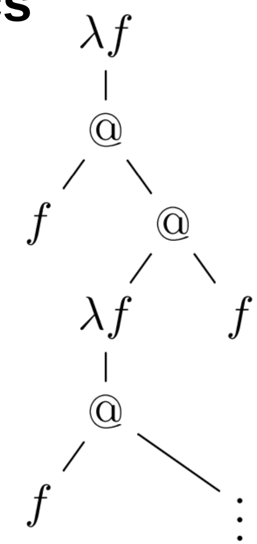
What is a higher-order program scheme?

$$Y = \lambda f. f(Y f)$$



Uninterpreted Semantics

$Y^\dagger =$



Interpreted Semantics

$$D \lhd \mathsf{CPO}(D, D)$$

Scott model of λ-calculus

$$Y^\dagger = \mathsf{fix} \in D \lhd \mathsf{CPO}(\mathsf{CPO}(D,D),D)$$

In general: signature Σ of givens

$$p_1 = f_1$$
$$\vdots$$
$$p_n = f_n$$

f_i are λ-Σ-terms over $p_1, \ldots, p_n$

Goal: Category-theoretic account of both semantics of ho-rps.

History & Background

W. Damm (1979): Higher-order program schemes and their languages

K. Aehlig (2006), Miranda (2006): renewed interest in ho-rps

M. Fiore, G. Plotkin, D. Turi (1999): Abstract syntax and variable binding
→ Substitution for finite λ-terms

R. Matthes, T. Uustalu (2004): Substitution in non-well founded syntax with variable binding

AMV (2006): Iterative algebras at work
→ category-theoretic treatment of 1st-order recursion

L. Moss, SM (2006): The category-theoretic solution of recursive program schemes

Combine these ideas for an abstract semantics of higher-order program schemes.

Outline

1. (Completely) iterative algebras
2. Finite and infinite λ-trees categorically.
3. Abstract uninterpreted semantics of ho-rps.
4. Interpreted semantics of abstract ho-rps in Scott models.

Iterative algebras – 1st-order case

Σ signature; Iterative Σ-algebra = Σ-algebra A such that finite systems

$$\begin{aligned} x_1 &= t_1(x_1, \ldots, x_n, y_1, \ldots, y_p) \\ &\vdots \\ x_n &= t_n(x_1, \ldots, x_n, y_1, \ldots, y_p) \end{aligned}$$

$t_i = \sigma(x_{i_1}, \ldots, x_{i_n}), \quad \sigma \in \Sigma_n$
or
$t_i \in A.$

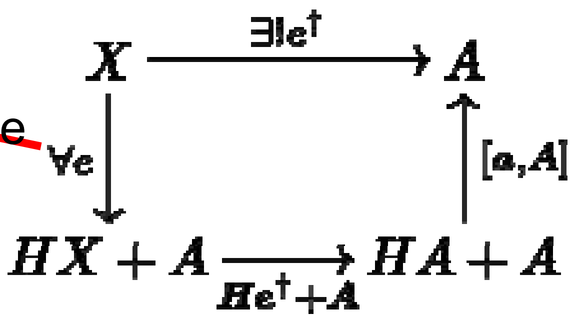
have a unique solution in A.

completely iterative algebra = finite and infinite systems have unique solutions

Categorically: take $HX = H_\Sigma X = \coprod_{n \in N} \Sigma_n \bullet X^n$ on Set

Σ-algebra → $HA \xrightarrow{a} A$

A completely A iterative → ~~X finite~~
iterative (cia)



Theorem. TFAE:

1. T is a final H-coalgebra,
2. T is an initial cia for H.

Category-theoretic setting: presheaves

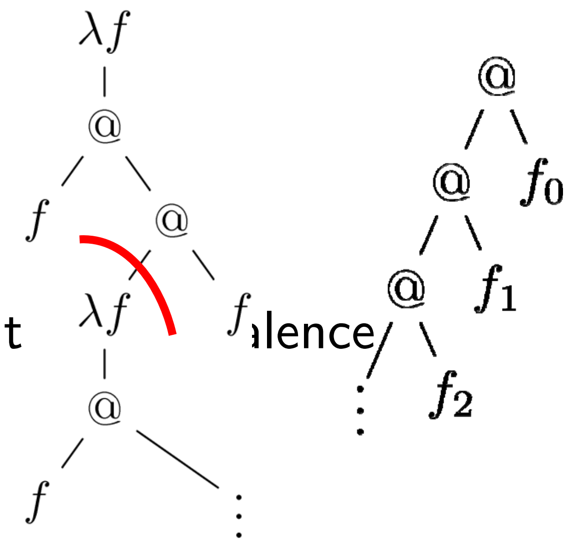
$\mathcal{F}$ = category of finite sets and all maps $\mathsf{Set}^{\mathcal{F}}$ = sets in context

$X : \mathcal{F} \longrightarrow \mathsf{Set}$ $X(\Gamma)$ = „terms“ in context Γ

Examples Σ signature of givens

$$\left.\begin{aligned} V(\Gamma) &= \Gamma \text{ presheaf of variables} \\ F_{\lambda,\Sigma}(\Gamma) &= \textit{finite } \lambda\text{-}\Sigma\text{-trees on } \Gamma \\ R_{\lambda,\Sigma}(\Gamma) &= \textit{rational } \lambda\text{-}\Sigma\text{-trees on } \Gamma \\ T_{\lambda,\Sigma}(\Gamma) &= \textit{all } \lambda\text{-}\Sigma\text{-trees on } \Gamma \end{aligned}\right\}$$

up t … alence



λ-terms as free algebras for endofunctors

$$t ::= x \mid tt \mid \lambda x.t \mid \sigma(t,\dots,t)$$

$$H_\lambda H_{\lambda,\Sigma} : \mathsf{Set}^{\mathcal{F}} \longrightarrow \mathsf{Set}^{\mathcal{F}}$$

$$H_\lambda X H_{\lambda,\Sigma} X = H_\lambda X + \coprod_{n\in\mathbb{N}} \Sigma_n \bullet X^n$$

$$\delta X(\Gamma) = X(\Gamma + 1)$$

Where are the variables?

$$\lambda x.t \leftarrow n+1 \text{ variables}$$

$$H_{\lambda,\Sigma}(V) \quad \text{...ables}$$

Theorem. $F_{\lambda,\Sigma}$ is the free $H_{\lambda,\Sigma}$-algebra on V (Fiore, Plotkin, Turi)

$R_{\lambda,\Sigma}$ is the free iterative $H_{\lambda,\Sigma}$-algebra on V

$T_{\lambda,\Sigma}$ is the free completely iterative $H_{\lambda,\Sigma}$-algebra on V

But this is not good enough …

What about substitution and ho-rps solutions?

$(\mathsf{Set}^{\mathcal{F}}, \otimes, V)$ monoidal category $\cong$ $(\mathsf{Fin}(\mathsf{Set}), \otimes, \mathrm{Id})$

$$X \otimes Y(\Gamma) = \int^{\bar{\Gamma}} \mathsf{Set}(\bar{\Gamma}, Y(\Gamma)) \bullet X(\bar{\Gamma}) \qquad X \otimes Y = X \cdot Y$$

simultaneous substitution of Y-term into X-terms

$F_{\lambda,\Sigma}$ $R_{\lambda,\Sigma}$ $T_{\lambda,\Sigma}$ are monoids = terms closed under simultaneous substitution

Important: the monoid structure needs to be compatible with the algebraic structure.

H-Monoids

Definition. (A, a, m, i) $H_{\lambda,\Sigma}$-monoid iff $H_{\lambda,\Sigma} A \xrightarrow{a} A$ algebra ← (completely) iterative

$A \otimes A \xrightarrow{m} A \xleftarrow{i} I$ monoid

(completely) iterative → $H_{\lambda,\Sigma}$-monoid

such that

point strength → $s_{(A,i),(A,i)}$

$$\begin{array}{ccccc} H_{\lambda,\Sigma} A \otimes A & \xrightarrow{s_{(A,i),(A,i)}} & H_{\lambda,\Sigma}(A \otimes A) & \xrightarrow{Hm} & H_{\lambda,\Sigma} A \\ \downarrow{\scriptstyle a \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle a} \\ A \otimes A & & \xrightarrow[m]{\qquad\qquad} & & A \end{array}$$

Theorem. (Fiore, Plotkin, Turi)

$F_{\lambda,\Sigma}$ = initial $H_{\lambda,\Sigma}$-monoid.

$R_{\lambda,\Sigma}$ = initial iterative $H_{\lambda,\Sigma}$-monoid.

$T_{\lambda,\Sigma}$ = initial completely iterative $H_{\lambda,\Sigma}$-monoid.
(partially by Matthes + Uustalu)

Higher-Order Recursion Schemes

Definition.

finitely presentable

higher-order recursion scheme = $X \xrightarrow{e} F_{\lambda,\Sigma} \otimes (X + V)$

$\mathsf{Set}^{\mathcal{F}}$ is a locally finitely presentable category → „good“ notion of finite objects

finitely presentable presheaves: $X_\Sigma(\Gamma) = \coprod_{n \in \mathbb{N}} \Sigma_n \bullet X^n$ and quotients

finite signature

uninterpreted solution of a recursion scheme:

embedding $j : F_{\lambda,\Sigma} \to R_{\lambda,\Sigma}$

$$\begin{array}{ccc}
X & \xrightarrow{\quad e^\dagger \quad} & R_{\lambda,\Sigma} \\
{\scriptstyle e}\downarrow & & \uparrow{\scriptstyle m^R} \\
F_{\lambda,\Sigma} \otimes (X+V) & & \\
{\scriptstyle j \otimes (X+V)}\downarrow & & \\
R_{\lambda,\Sigma} \otimes (X+V) & \xrightarrow[R_{\lambda,\Sigma} \otimes [e^\dagger, i^R]]{} & R_{\lambda,\Sigma} \otimes R_{\lambda,\Sigma}
\end{array}$$

Unique Uninterpreted Solutions

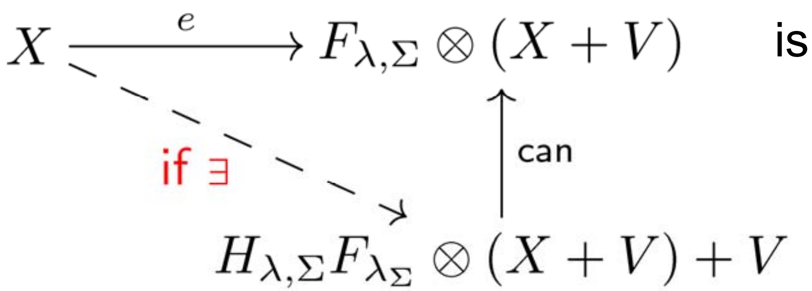
Problem. $p_1 = p_1$ has no unique solution

Solution. disallow single variables on the right-hand sides of equations:

$$F_{\lambda,\Sigma} = \text{free } H_{\lambda,\Sigma}\text{-algebra on V} \quad \Longrightarrow \quad F_{\lambda,\Sigma} = H_{\lambda,\Sigma} F_{\lambda,\Sigma} + V$$

$$\begin{aligned} F_{\lambda,\Sigma} \otimes (X + V) &\cong (H_{\lambda,\Sigma} F_{\lambda,\Sigma} + V) \otimes (X + V) \\ &\cong H_{\lambda,\Sigma} F_{\lambda,\Sigma} \otimes (X + V) + (\not{X} + V) \end{aligned}$$

Definition.



is guarded

Theorem. Every guarded recursion scheme has a unique uninterpreted solution.

Interpreted Solutions

Given: Scott model of λ-calculus, i.e.,

- embedding-projection pair → $\mathsf{fold} : \mathsf{CPO}(D, D) \lhd D : \mathsf{unfold}$
- continuous operations → $\sigma^D : D^n \longrightarrow D, \quad \sigma \in \Sigma_n$

This gives a presheaf → $\langle D, D\rangle : \mathcal{F} \longrightarrow \mathsf{Set}, \quad \langle D, D\rangle(\Gamma) = \mathsf{CPO}(D^\Gamma, D)$

Proposition. $\langle D, D\rangle$ is an $H_{\lambda,\Sigma}$-monoid (Fiore, Plotkin, Turi)

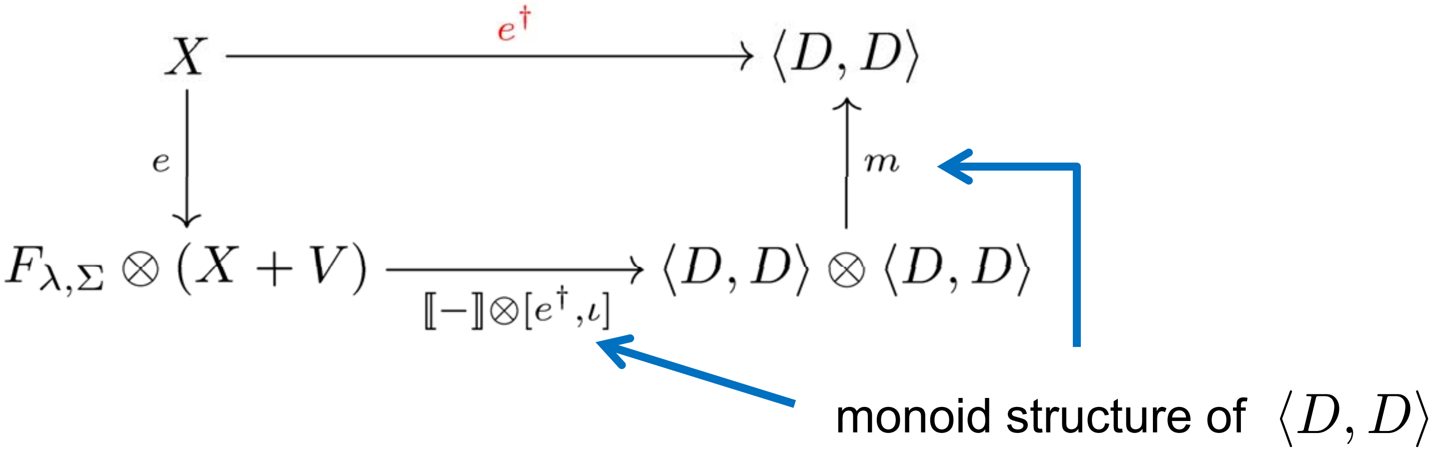
Notation. $[\![-]\!] : F_{\lambda,\Sigma} \longrightarrow \langle D, D\rangle$ unique $H_{\lambda,\Sigma}$-monoid homomorphism.

$$F_{\lambda,\Sigma}(\Gamma) \xrightarrow{[\![-]\!]_\Gamma} \langle D, D\rangle(\Gamma) = \mathsf{CPO}(D^\Gamma, D)$$
$$t \longmapsto [\![t]\!]_\Gamma : D^\Gamma \longrightarrow D$$

↑ continuous function evaluating the term t

Interpreted Solutions (2)

Definition. interpreted solution of $X \xrightarrow{e} F_{\lambda,\Sigma} \otimes (X + V)$ in D:



Theorem. Every higher-order recursion scheme has a least interpreted solution in D.

Conclusions + Future

We combined the work of Fiore, Plotkin & Turi on abstract variable binding and of AMV on iterative algebras for a categorical treatment of higher-order recursion scheme semantics.

Higher-order recursion schemes have:

- unique uninterpreted solutions,
- least interpreted solutions in Scott models of λ-calculus.

Future:

- Typed λ-calculus
- Relation of uninterpreted and interpreted semantics (Mezei-Wright-Theorem)
- Sound and complete logical calculi for the equivalence of recursion schemes
- Other applications